PMATH450

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Contents

1 Measure

1.1 Borel Set

Definition 1

X is a set. We call $a \subseteq \mathcal{P}(x)$ a <u>σ-algebra</u> of subsets of X if:

1. $\emptyset \in a$
2. $A \in a \implies X \setminus A \in a$
3. $A_1, A_2, A_3, \ldots, \in a \implies \bigcup_{i=1}^{\infty} A_i \in a$

Remark. $a \subseteq \mathcal{P}(X)$ is a σ-algebra

1. $X \in a,\ X \setminus \emptyset = X \in a$
2. $A, B \in a \implies A \bigcup B \in a$ by $A \bigcup = A \bigcup B \bigcup \underbrace{\emptyset \ldots \bigcup \emptyset \ldots}_{\text{countably many}} \in a$
3. $A_1, A_2, \ldots \in a \implies \bigcap_{i=1}^{\infty} A_i \in a$, by $\bigcap_{i=1}^{\infty} A_i = X \setminus \left(\bigcup_{i=1}^{\infty} (X \setminus A_i) \right) \in a$
4. $A, B \in a \implies A \bigcap B \in a$

Example 1: σ-algebra

- $\{\emptyset, X\}$
- $a = \mathcal{P}(x)$
- $a = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ is not a σ-algebra. $A = (0, 1) \in a$, but $\mathbb{R} \setminus A = (-\infty, 0] \cup [1, \infty) \notin a$ because it's not open
- $a = \{A \subseteq \mathbb{R} : A \text{ is open or closed}\}$ is not a σ-algebra, because $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \notin a$ (Q is countable)

Proposition 1

X is a set, $C \subseteq \mathcal{P}(x)$, then

$$a := \bigcap \{\mathbb{B} : \mathbb{B}\ \sigma\text{-algebra},\ C \subseteq \mathbb{B}\} \text{ is a } \sigma\text{-algebra}$$

It's the smallest σ-algebra containing C.

Definition 2

$C = \{A \subseteq \mathbb{R} : A \text{ open}\}$, then

$$a = \cap\{\mathbb{B} : C \subseteq \mathbb{B}, \mathbb{B}\ \sigma - \text{algebra}\}$$

is a Borel σ-algebra. The elements of a are called the Borel Sets.

Remark.

1. open $\implies$ Borel
2. closed $\implies$ Borel
3. $\{X_1, X_2, \dots\} = \bigcup_{i=1}^{\infty} \{X_i\}$, so countable $\implies$ Borel. (Note $\mathbb{Q}$ is not open or closed but Borel)
4. $[a, b) = [a, b] \setminus \{b\} = [a, b] \cap (\mathbb{R} \setminus \{b\})$, so a half open interval is also Borel

1.2 Outer Measure

Goal: Define a function

$$m : \mathcal{P}(\mathbb{R}) \mapsto [0, \infty) \cup \{\infty\} \text{ (called a measure)}$$

1. $m((a, b)) = m([a, b]) = m((a, b]) = b - a$
2. $m(A \cup B) \leqslant m(A) + m(B)$
3. $A \cap B = \emptyset$, $m(A \cup B) = m(A) + m(B)$

Definition 3

We define a (Lebesgue) <u>outer measure</u> by

$$\begin{aligned} &m^* : \mathcal{P}(\mathbb{R}) \mapsto [0, \infty) \cup \{\infty\} \\ &m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i,\ I_i \text{ open, bounded interval} \right\} \end{aligned}$$

Example 2

$\emptyset \implies m^*(\emptyset) = 0$. Since $\forall \varepsilon > 0,\ \emptyset \subseteq (0, \varepsilon) \implies m^*(\emptyset) \leqslant l((0, \varepsilon))$. Since $m^*(\emptyset) \geqslant 0$, we know $m^*(\emptyset) = 0$

Example 3

$A = \{x_1, x_2, \ldots\}$ is countable, then

$$A \subseteq \bigcup_{i=1}^{\infty} \left(x_i - \frac{\varepsilon}{2^{i+1}}, x_i + \frac{\varepsilon}{2^{i+1}} \right),\ \varepsilon > 0$$

then

$$\begin{aligned} m^*(A) &\leqslant \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \\ &= \frac{\varepsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \\ &= \frac{\varepsilon}{2} \left(\frac{1}{1 - 1/2} \right) = \varepsilon \end{aligned}$$

Since ε is arbitrary,

$$m^*(A) = 0$$

It's also clear that finite set also have measure 0. That is, both countable and finite sets have measure 0

1.3 Outer Measure 2

Proposition 2

If $A \subseteq B$, then $m^*(A) \leqslant m^*(B)$

Proof.

$$\begin{aligned} X &:= \left\{ \sum l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i \right\} \\ Y &:= \left\{ \sum l(I_i) : B \subseteq \bigcup_{i=1}^{\infty} I_i \right\} \\ Y &\subseteq X \\ \inf X &\leqslant \inf Y \end{aligned}$$

□

Lemma 3

If $a, b \in \mathbb{R}$ with $a \leqslant b$, then $m^*([a,b]) = b - a$

Proof. Let $\varepsilon > 0$ be given. Since $[a,b] \subseteq \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right)$. We see that $m^*([a,b]) \leqslant b - a + \varepsilon$. Let I_i be bounded, open intervals such that $[a,b] \subseteq \bigcup\limits_{i=1}^{\infty} I_i$. Since $[a,b]$ is compact, then there exists $n \in \mathbb{N}$, such that

$$[a,b] \subseteq \bigcup_{i=1}^{n} I_i$$

so

$$b - a \leqslant \sum_{i=1}^{n} l(I_i) \leqslant \sum_{i=1}^{\infty} l(I_i)$$

and so $m^*([a,b]) \geqslant b - a \implies m^*([a,b]) = b - a$. Note $m^*([a,b]) > 0$ because of the definition of $\inf$. □

Proposition 4

If I is an interval, then $m^*(I) = l(I)$

Proof.

1. If I is bounded with endpoints $a \leqslant b$, then

$$\begin{aligned} \varepsilon > 0, I \subseteq [a,b] &\implies m^*(I) \leqslant m^*([a,b]) = b-a \\ \left[a+\frac{\varepsilon}{2}, b-\frac{\varepsilon}{2}\right] \subseteq I &\implies b-a+\varepsilon \leqslant m^*(I) \\ &\implies b-a \leqslant m^*(I) \end{aligned}$$

   then $m^*(I) = b-a$

2. If I is unbounded

$$\begin{aligned} &\forall n \in \mathbb{N}, \exists I_n, l(I_n) = n \\ \implies &m^*(I) \geqslant m^*(I_n) = n \\ \implies &m^*(I) = \infty = l(I) \end{aligned}$$

$\square$

1.4 Basic Properties of Outer Measure

Outer measure is

1. Translation Invariant
2. Countably Subadditive

Notation: $x \in \mathbb{R}, A \subseteq \mathbb{R}, x + A := \{x + a : a \in A\}$

Proposition 5: Translation Invariant

$m^*(x + A) = m^*(A)$

Proof.

$$\begin{aligned}
m^*(x + A) &= \inf\left\{\sum_{i=1}^{\infty} l(I_i) : x + A \subseteq \bigcup_{i=1}^{\infty} I_i, \text{ bounded, open}\right\} \\
&= \inf\left\{\sum_{i=1}^{\infty} l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i - x, \text{ bounded, open}\right\} \\
&= \inf\left\{\sum_{i=1}^{\infty} l(\underbrace{I_i - x}_{J_i}) : A \subseteq \bigcup_{i=1}^{\infty} \underbrace{I_i - x}_{J_i}, \text{ bounded, open}\right\} \\
&= \inf\left\{\sum_{i=1}^{\infty} l(J_i) : A \subseteq \bigcup_{i=1}^{\infty} J_i\right\} \\
&= m^*(A)
\end{aligned}$$

□

Proposition 6: Countably Subadditivity

If $A_i \subseteq \mathbb{R} (i \in \mathbb{N})$, then

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \bigcup_{i=1}^{\infty} m^*(A_i)$$

Proof. We may assume each $m^*(A_i) < \infty$(otherwise it's trivial). Let $\varepsilon > 0$ be given and let's fix $i \in \mathbb{N}$. There exists open and bounded interval $I_{i,j}$ such that $A_i \subseteq \bigcup\limits_{i=1}^{\infty} I_{i,j}$ and

$$\sum_{i=1}^{\infty} l(I_{i,j}) \leqslant m^*(A_i) + \frac{\varepsilon}{2^i}$$

We see that

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j} I_{i,j}$$

and so

$$
\begin{aligned}
m^*\left(\bigcup_{i=1}^{\infty}\right) &\leqslant \sum_{i,j} l(I_{i,j}) \\
&\leqslant \sum_{i=1}^{\infty}\left(m^*(A_i)+\frac{\varepsilon}{2^i}\right) \\
&= \sum_{i=1}^{\infty} m^*(A_i)+\sum_{i=1}^{\infty}\frac{\varepsilon}{2^i} \\
&= \sum_{i=1}^{\infty} m^*(A_i)+\varepsilon
\end{aligned}
$$

$\square$

Corollary 7: finite subadditivity

If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R})$, then

$$m^*(A_1 \cup A_2 \ldots \cup A_n) \leqslant m^*(A_1) + m^*(A_2) + \ldots + m^*(A_n)$$

Later we will see that there exists $A, B \subseteq \mathbb{R}$, $A \cap B = \emptyset$ but $m^*(A \cup B) \leqslant m^*(A) + m^*(B)$, we will solve this by restricting the domain of m^* to only include the sets which measure ”nicely”.

1.5 Measurable Sets

Definition 4

We say $A \subseteq \mathbb{R}$ is <u>measurable</u> if $\forall X \subseteq \mathbb{R}$,

$$m^*(X) = m^*(X \cap A) + m^*(X \setminus A)$$

Remark. Always have

$$m^*(X) \leqslant m^*(X \cap A) + m^*(X \setminus A)$$

by $X = (X \setminus A) \cup (X \cap A)$

Remark. If $A \subseteq \mathbb{R}$ is measurable and $B \subseteq \mathbb{R}$ with $A \cap B = \emptyset$, then

$$m^*(\underbrace{A \cup B}_{X}) = m^*(X \cap A) + m^*(X \setminus A) = m^*(A) + m^*(B)$$

Proposition 8

If $m^*(A) = 0$, then A is measurable

Proof. Let $X \subseteq \mathbb{R}$, since $X \cap A \subseteq A$, we have

$$0 \leqslant m^*(X \cap A) \leqslant m^*(A) = 0$$

so $m^*(X \cap A) = 0$, then

$$\begin{aligned} & m^*(X \cap A) + m^*(X \setminus A) \\ =& m^*(X \setminus A) \\ \leqslant & m^*(X) \end{aligned}$$

the other direction is always true, so

$$m^*(X) = m^*(X \cap A) + m^*(X \setminus A)$$

□

Proposition 9

$A_1, \ldots, A_n$ measurable, then $\bigcup_{i=1}^{n} A_i$ is measurable.

Proof. It suffices to prove the result when $n = 2$.
Let $A, B \subseteq \mathbb{R}$ be measurable. Let $X \subseteq \mathbb{R}$, then

$$\begin{aligned} m^*(X) &= m^*(X \cap A) + m^*(\underbrace{X \setminus A}_{Y}) \\ &= m^*(X \cap A) + m^*(Y \cap B) + m^*(Y \setminus B) \\ &= m^*(X \cap A) + m^*((X \setminus A) \cap B) + m^*(X \setminus (A \cup B)) \\ &\geqslant m^*((X \cap A) \cup ((X \setminus A) \cap B)) + m^*(X \setminus (A \cup B)) \\ &= m^*(X \cap (A \cup B)) + m^*(X \setminus (A \cup B)) \end{aligned}$$

□

Proposition 10

$A_1, A_2, \ldots, A_n$ measurable, $A_i \cap A_j = \emptyset, i \neq j$. Let $A = A_1 \cup \ldots \cup A_n$. If $X \subseteq \mathbb{R}$, then

$$m^*(X \cap A) = \sum_{i=1}^{n} m^*(X \cap A_i)$$

Proof. For $n = 2$, let $A, B \subseteq \mathbb{R}$ measurable, $A \cap B = \emptyset$. Let $X \subseteq \mathbb{R}$, then

$$\begin{aligned} &m^*(X \cap (A \cup B)) \\ =&m^*((X \cap (A \cup B)) \cap A) + m^*((X \cap (A \cup B)) \setminus A) \\ =&m^*(X \cap A) + m^*(X \cap B) \end{aligned}$$

Note: we only need $n - 1$ sets to be measurable, it's ok if one set is not. □

Corollary 11: Finite Additive

$A_1, \ldots, A_n$ measurable, $A_i \cap A_j = \emptyset$, then $m^*(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^{n} m^*(A_i)$

Proof. Take $X = \mathbb{R}$, use the proposition above. □

1.6 Countably Additivity

Lemma 12

$A_i \subseteq \mathbb{R}$ measurable ($i \in \mathbb{N}$). If $A_i \cap A_j = \emptyset$ for $i \neq j$, then $A := \bigcup_{i=1}^{\infty} A_i$ is measurable.

Proof. Let $B_n = A_1 \cup \ldots A_n$ and $X \subseteq \mathbb{R}$ arbitrary.

$$\begin{aligned} m^*(X) &= m^*(X \cap B_n) + m^*(X \setminus B_n) \\ &\geqslant m^*(X \cap B_n) + m^*(X \setminus A) \\ &= \sum_{i=1}^{m} m^*(X \cap A_i) + m^*(X \setminus A) \end{aligned}$$

Taking $n \to \infty$,

$$\begin{aligned} m^*(X) &\geqslant \sum_{i=1}^{\infty} m^*(X \cap A_i) + m^*(X \setminus A) \\ &= m^*\left(\bigcup_{i=1}^{\infty}(X \cap A_i)\right) + m^*(X \setminus A) \\ &= m^*(X \cap A) + m^*(X \setminus A) \end{aligned}$$

□

Proposition 13

$A \subseteq \mathbb{R}$ measurbale, then $\mathbb{R} \setminus A$ is measurable.

Proof. $X \subseteq \mathbb{R}$,

$$\begin{aligned} &m^*(X \cap (\mathbb{R} \setminus A)) + m^*(X \setminus (\mathbb{R} \setminus A)) \\ =&m^*(X \setminus A) + m^*(X \cap A) \\ =&m^*(X) \text{ by } A \text{ measurable} \end{aligned}$$

□

Proposition 14

$A_i \subseteq \mathbb{R}$ measurable ($i \in \mathbb{N}$), then $A = \bigcup_{i=1}^{\infty} A_i$ is measurable.

Proof. $B_n = A_n \setminus (A_1 \cup \ldots \cup A_{n-1}) = A_n \cap (\mathbb{R} \setminus (A_1 \cup \ldots \cup A_{n-1}))$, $(B_1 = A_1)$, $n \geqslant 2$, we can see that B_n is an intersection of measurable sets, hence measurable. And, for $i \neq j$, $B_i \cap B_j = \emptyset$. Also,

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

so A is measurable by lemma above. □

Corollary 15

The collection $\mathcal{L}$ of (Lebesgue) measurable sets is a σ-algebra of sets in $\mathbb{R}$

Proposition 16: Countably Additivity

$A_i \subseteq \mathbb{R}$ measurable ($i \in \mathbb{N}$), if $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$$

Proof.

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geqslant m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{=1}^{\infty} m^*(A_i)$$

Take $n \to \infty$, then

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geqslant \sum_{i=1}^{\infty} m^*(A_i)$$

The other direction follows by the subadditivity. □

1.7 Measurable Sets Continued

Proposition 17: I

$a \in \mathbb{R}$, then (a, ∞) is measurable

Proof. Let $X \subseteq \mathbb{R}$. We want to show that

$$m^*(X \cap (a, \infty)) + m^*(X \setminus (a, \infty)) \leqslant m^*(X)$$

1. $a \notin X$,
   We show
   $$m^*(\underbrace{X \cap (a, \infty)}_{X_1}) + m^*(\underbrace{X \cap (-\infty, a)}_{X_2}) \leqslant m^*(X)$$
   Let (I_i) be a sequence of bounded, open intervals such that $X \subseteq \bigcup I_i$. Define
   $$I_i' = I_i \cap (a, \infty) \text{ and } I_i'' = I_i \cap (-\infty, a)$$
   Note that
   $$X_1 \subseteq \bigcup I_i', X_2 \subseteq \bigcup I_i''$$
   and so
   $$\begin{aligned} m^*(X_1) &\leqslant \sum l(I_i') \\ m^*(X_2) &\leqslant \sum l(I_i'') \end{aligned}$$
   We then see that
   $$\begin{aligned} &m^*(X_1) + m^*(X_2) \\ \leqslant &\sum l(I_i') + \sum l(I_i'') \\ = &\sum (l(I_i') + l(I_i'')) \\ = &\sum l(I_i) \end{aligned}$$
   By the definition of $\inf$, we have
   $$m^*(X_1) + m^*(X_2) \leqslant m^*(X)$$
2. $a \in X$, let $X' = X \setminus \{a\}$, then
   $$\begin{aligned} m^*(X \cap (a, \infty)) + m^*(X \setminus (a, \infty)) &= m^*((X' \cup \{a\}) \cap (a, \infty)) + m^*((X' \cup \{a\}) \setminus (a, \infty)) \\ &= m^*(X' \cap (a, \infty)) + m^*((X' \setminus (a, \infty)) \cup \{a\}) \\ &\leqslant m^*(X' \cap (a, \infty)) + m^*(X' \setminus (a, \infty)) + m^*(\{a\}) \\ &= m^*(X') + 0 \leqslant m^*(X) \end{aligned}$$

The other direction is trivial by subadditivity. □

Theorem 18

Borel set is measurable

Proof. (a, ∞) is measurable, so $\bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, \infty\right) = [a, \infty)$ is measurable. So $\mathbb{R} \setminus [a, \infty) = (-\infty, a)$ is measurable, then $(a, b) = (a, \infty) \cap (-\infty, b)$ is measurable. Hence, every open set in $\mathbb{R}$ is measurable (open sets can be expressed as countable union of open intervals), so

$$\mathbb{B} \subseteq \mathcal{L}$$

because $\mathbb{B}$ is the smallest σ-algebra containing all open sets and $\mathcal{L}$ is a σ-algebra containing all open sets. □

Definition 5

We call $m : \mathcal{L} \mapsto [0, \infty) \cup \{\infty\}$ given by $m(A) = m^*(A)$, the Lebesgue Measure

Remark. $A \subseteq \mathbb{R}$ measurable, then $x + A$ is measurable $\forall x \in \mathbb{R}$

Proof. $\forall K \subseteq \mathbb{R},\ K - x \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(K - x) &= m^*(A \cap (K - x)) + m^*(A \setminus (K - x)) \\ &= m^*((A + x) \cap K) + m^*((A + x) \setminus K) \\ &= m^*(K) \end{aligned}$$

□

1.8 Basic Properties of Lebesgue Measure

Proposition 19: Excision Properties

$A \subseteq B$, A measurable, $m(A) < \infty$, then $m^*(B \setminus A) = m^*(B) - m(A)$

Proof.

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \setminus A) \\ &= m^*(A) + m^*(B \setminus A) \\ &= \underbrace{m(A)}_{<\infty} + m^*(B \setminus A) \end{aligned}$$

□

Theorem 20: Continuity of Measure

1. $A_1 \subseteq A_2 \subseteq A_3 \ldots$, measurable, then

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} m(A_n)$$

2. $B_1 \supseteq B_2 \supseteq B_3 \ldots$, measurable, and $m(B_1) < \infty$, then

$$m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$$

Proof.

1. Since $m(A_k) \leqslant m(\cup A_i),\ \forall k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} m(A_n) \leqslant m(\cup A_i)$$

   if $\exists k \in \mathbb{N}$ such that $m(A_k) = \infty$, then $\lim_{n\to\infty} m(A_n) = \infty$ and we are done, so assume $m(A_k) < \infty,\ \forall k \in \mathbb{N}$.
   For each $k \in \mathbb{N}$, let $D_k = A_k \setminus A_{k-1}$, $A_0 \neq \emptyset$. Note

   - D_k's are measurable
   - D_k's are parwise disjoint
   - $\cup D_i = \cup A_i$

so

$$\begin{aligned}
m^*(\cup A_i) &= m^*(\cup D_i) \\
&= \sum_{i=1}^{\infty} m(D_i) \\
&= \sum_{i=1}^{\infty} m(A_i) - m(A_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} m(A_i) - m(A_{i-1}) \\
&= \lim_{n\to\infty} m(A_n) - m(A_0) \\
&= \lim_{n\to\infty} m(A_n)
\end{aligned}$$

2. For $k \in \mathbb{N}$, define

$$D_k = B_1 \setminus B_k$$

Note:

- D_k's measurable
- $D_1 \subseteq D_2 \subseteq D_3 \subseteq \dots$

By 1), we know $m(\cup D_i) = \lim_{n\to\infty} m(D_n)$, we see that

$$\cup D_i = \bigcup_{i=1}^{\infty} (B_1 \setminus B_i) = B_1 \setminus \left(\bigcap_{i=1}^{\infty} B_i \right)$$

and so,

$$\lim_{n\to\infty} m(D_n) = m(\cup D_i) = m(B_1 \setminus (\cap B_i)) = m(B_1) - m(\cap B_i)$$

because $\cap B_i$ is measurable and has finite measure.
However,

$$\begin{aligned}
\lim_{n\to\infty} m(D_n) &= \lim_{n\to\infty} m(B_1 \setminus B_n) \\
&= \lim_{n\to\infty} m(B_1) - m(B_n) \\
&= m(B_1) - \lim_{n\to\infty} m(B_n) \\
&= m(B_1) - m(\cap B_i)
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} m(B_n) = m(\cap B_i)$$

$\square$

Example 4

$B_i = (i, \infty)$, and $m(\cap B_i) = m(\emptyset) = 0$, but $\lim_{n\to\infty} m(B_n) = \infty$

1.9 Non-Measurable Sets

Lemma 21

$A \subseteq \mathbb{R}$ bounded, measurable $\Lambda \subseteq \mathbb{R}$ bounded, countably infinite. If $\lambda + A$, $\lambda \in \Lambda$ are pairwise disjoint, then $m(A) = 0$

Proof. $\bigcup\limits_{\lambda \in \Lambda} (\lambda + A)$ is a bounded set, which is measurable, then

$$m\left(\bigcup_{\lambda}(\lambda + A)\right) < \infty$$
$$m\left(\bigcup_{\lambda}(\lambda + A)\right) = \sum_{\lambda} m(\lambda + A) = \sum_{\lambda} m(A) < \infty$$

and $m(A) \geqslant 0$, so $m(A) = 0$ (Λ is countably infinite) $\square$

Construction: Start with $\emptyset \neq A \subseteq \mathbb{R}$, consider $a \sim b \iff a - b \in \mathbb{R}$. Then $\sim$ is an equivalence relation.
Let C_A denotes a single choice of equivalence class representatives for A relative to $\sim$.

Remark. The sets $\lambda + C_A$, $\lambda \in \mathbb{Q}$ are pairwise disjoint

Proof. say $x \in (\lambda_1 + C_A) \cap (\lambda_2 \cap C_A)$

$$\begin{aligned} & x = \lambda_1 + a = \lambda_2 + b \\ \implies & a, b \in C_A \\ \implies & a - b = \lambda_1 - \lambda_2 \in \mathbb{Q} \\ \implies & a \sim b \implies a = b \text{ by each equiv. class has one repre.} \\ \implies & \lambda_1 = \lambda_2 \end{aligned}$$

$\square$

Theorem 22: Vitali

Every set $A \subseteq \mathbb{R}$ with $m^*(A) > 0$ contains a non-measurable subset.

Proof. By Quiz1, we may assume A is bounded, say $A \subseteq [-N, N]$, for some $N \in \mathbb{N}$.

<u>Claim:</u> C_A is non-measurable.
Assume C_A is measurable. Let $\Lambda \subseteq \mathbb{Q}$ be bounded, infinite. By the lemma and remark,

$$m(C_A) = 0$$

Let $a \in A$, then $a \sim b$ for some $b \in C_A$. In particular, $a - b = \lambda \in \mathbb{Q}$. Moreover,

$$\lambda \in [-2N, 2N]$$

Taking $\Lambda_0 = \mathbb{Q} \cap [-2N, 2N]$, have

$$A \subseteq \bigcup_{\lambda \in \Lambda_0} (\lambda + C_A)$$

so $m^*(A) = 0$, contradiction $\square$

Corollary 23

$\exists A, B \subseteq \mathbb{R}$, such that

1. $A \cap B = \emptyset$, and
2. $m^*(A \cup B) < m^*(A) + m^*(B)$

Proof. Let C be a non-measurable set, $\exists X \subseteq \mathbb{R}$ such that

$$m^*(X) < m^*(\underbrace{X \cap C}_{A}) + m^*(\underbrace{X \setminus C}_{B})$$

$\square$

1.10 Cantor-Lebesgue Function

Recall: Cantor Set

$$\begin{aligned} I &= [0,1] \\ C_1 &= [0,1/3] \cup [2/3,1] \\ C_2 &= [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1] \\ &\vdots \\ C &= \bigcap_{k=1}^{\infty} C_k \end{aligned}$$

Note C is <u>countable</u> and <u>closed</u>.

Proposition 24

The Cantor Set is Borel and has measure zero.

Proof. Closed $\implies$ Borel. And $C = \bigcap_{k=1}^{\infty} C_k$, where C_k's measurable and

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

By continuity of measure,

$$\begin{aligned} m(C) &= \lim_{k\to\infty} m(C_k) \\ &= \lim_{k\to\infty} \frac{2^k}{3^k} = 0 \end{aligned}$$

$\square$

Construction: Cantor-Lebesgue Function (C-L fcn)

1. For $k \in \mathbb{N}$, U_k= Union of open intervals deleted in the process of constructing $C_1, C_2, \ldots, C_k$ i.e. $U_k = [0,1] \setminus C_k$.

2. $U = \bigcup_{k=1}^{\infty} U_k$, i.e. $U = [0,1] \setminus C$

3. Say $U_k = I_{k,1} \cup I_{k,2} \cup \ldots \cup I_{k,2^k-1}$ (In order: from left to right). Define

$$\varphi : U_k \to [0,1] \text{ by } \varphi|_{I_{k,i}} = \frac{i}{2^k}$$

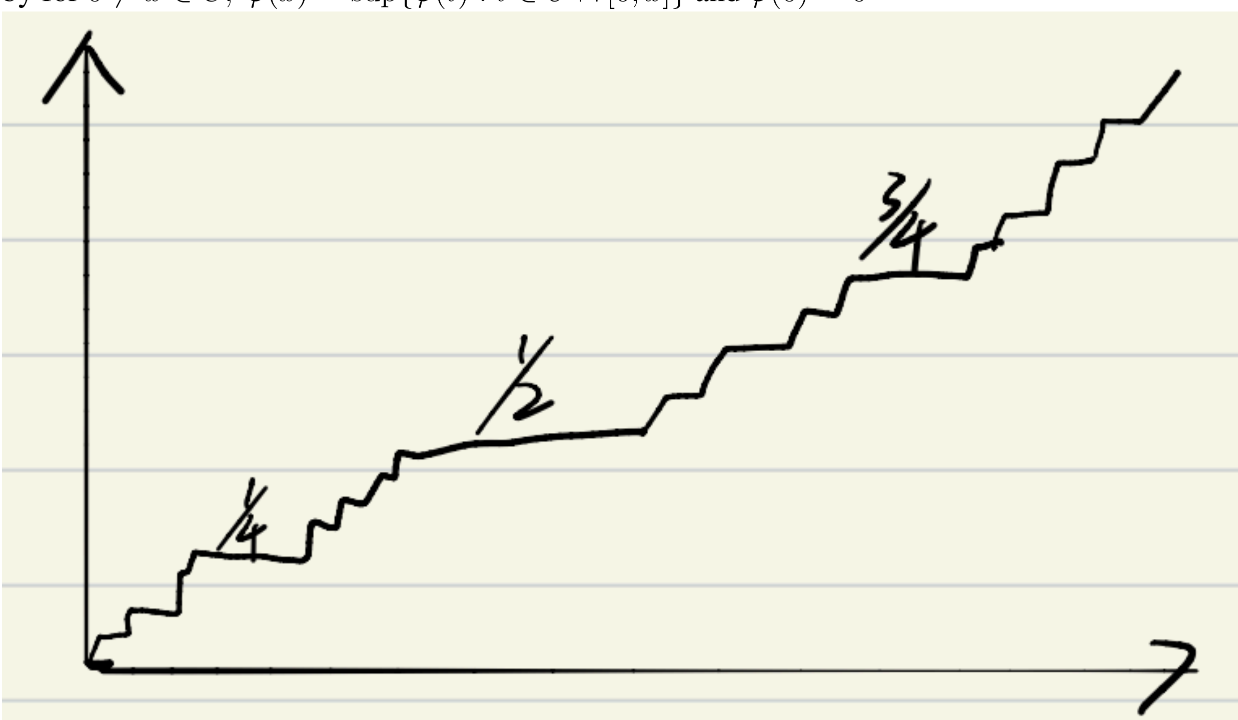
e.g. $U_1 = (1/3, 2/3) \to \frac{1}{2^1} = \frac{1}{2}$ and

$$\begin{aligned} U_2 = &(1/9, 2/9) & &\cup (1/3, 2/3) & &\cup (7/9, 8/9) \\ &\to \frac{1}{4} & &\to \frac{2}{4} & &\to \frac{3}{4} \end{aligned}$$

4. Define

$$\varphi : [0,1] \to [0,1]$$

by for $0 \neq x \in C$, $\varphi(x) = \sup\{\varphi(t) : t \in U \cap [0,x]\}$ and $\varphi(0) = 0$



Things to know about φ

1. φ is increasing. Take two points in U, for large enough k, both points in U_k. If they are in the Cantor Set, then it's increasing by definition

2. φ is continuous

   - φ is continuous on U. (It's constant on a small interval)
   - $x \in C,\ x \neq 0, 1$. For large k, $\exists a_k \in I_{k,i}$, $\exists b_k \in I_{k,i+1}$ such that

     $$a_k < x < b_k$$

     but, $\varphi(b_k) - \varphi(a_k) = \frac{i+1}{2^k} - \frac{i}{2^k} = \frac{1}{2^k} \to 0$
   - $x \in \{0, 1\}$

3. $\varphi : u \to [0, 1]$ is differentiable and $\varphi' = 0$

4. φ is onto,

   $$\varphi(0) = 0,\ \varphi(1) = 1$$

   by Intermediate Value Theorem.

1.11 A Non-Borel Set

Let φ be the Cantor-Lebesgue Function. Consider $\psi : [0,1] \to [0,2]$ defined by $\psi(x) = x + \varphi(x)$.

1. ψ is <u>strictly</u> increasing
2. ψ is continuous
3. ψ is onto

By 1),3), we know ψ is bijective, hence invertible.

<u>Properties:</u>

1. $\psi(C)$ is measurable and has <u>positive</u> measure.
2. ψ maps a particular (measurable) subset of C to a non-measurable set.

Proof.

1. By A1, ψ^{-1} is continuous, so $\psi(C) = (\psi^{-1})^{-1}(C)$ is closed, so $\psi(C)$ is Borel implies that it's measurable.
   Note that

   $$\begin{aligned} [0,1] &= C\dot{\cup}U \\ \implies [0,2] &= \psi(C\dot{\cup}U) = \psi(C)\dot{\cup}\psi(U) \text{ by bijectivity} \\ \implies 2 &= m(\psi(C)) + m(\psi(U)) \end{aligned}$$

   It suffices to show that

   $$m(\psi(U)) = 1$$

   Say $U = \dot{\bigcup}_{i=1}^{\infty} I_i$, where I_i are disjoint open intervals. Then

   $$\psi(U) = \dot{\bigcup_{i=1}^{\infty}} \psi(I_i) \implies m(\psi(U)) = \sum m(\psi(I_i))$$

   Note that $\forall i \in \mathbb{N},\ \exists r \in \mathbb{R}$, such that $\varphi(x) = r, \forall x \in I_i$
   In particular, $\psi(x) = x + r, \forall x \in I_i$ and so

   $$\psi(I_i) = r + I_i$$

   so

   $$m(\psi(U)) = \sum m(\psi(I_i)) = \sum m(I_i) = m(\dot{\cup} I_i) = m(U)$$

   Since $[0,1] = U\dot{\cup}C$, we have that $1 = m(U) + m(C) = m(U)$, so $m(\psi(U)) = m(U) = 1 > 0 \implies m(\psi(C)) = 1$

2. By Vitali, $\psi(C)$ contains a subset $A \subseteq \psi(C)$ which is non-measurable. Let $B = \psi^{-1}(A) \subseteq C$, B is measurable because $0 = m(C) \geqslant m(B) = 0$. Then $\psi(B) = \psi(\psi^{-1}(A)) = A$

$\square$

Theorem 25

Cantor Set contains an element $\mathcal{L} \setminus \mathbb{B}$

Proof. $B \subseteq C \implies B$ measurable. $\psi(B)$ is non-measurable. By A1, if B is Borel, then $\psi(B)$ is Borel, so B cannot be Borel. □

1.12 Measurable Function

Definition 6

$A \subseteq \mathbb{R}$ measurable, we say $f : A \to \mathbb{R}$ is measurable iff for all open $U \subseteq \mathbb{R}$, $f^{-1}(U)$ measurable.

Proposition 26

If $A \subseteq \mathbb{R}$ is measurable and $f : A \to \mathbb{R}$ is continuous then f is measurable.

Proof. f is continuous $\implies f^{-1}(U)$ open if U open $\implies f^{-1}(U)$ Borel, measurable ☐

Proposition 27

$A \subseteq \mathbb{R}$ measurable, $\chi_A : \mathbb{R} \to \mathbb{R}$, $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, then χ_A is measurable.

Proof.

$$\begin{aligned} &U \subseteq \mathbb{R}, \text{ open} \\ &\chi_A^{-1}(U) = \mathbb{R}, \text{ if } 0, 1 \in U \\ &\chi_A^{-1}(U) = A, \text{ if } 1 \in U, 0 \notin U \\ &\chi_A^{-1}(U) = A^C, \text{ if } 0 \in U, 1 \notin U \\ &\chi_A^{-1}(U) = \emptyset, \text{ if } 0, 1 \notin U \end{aligned}$$

In any case, $\chi_A^{-1}(U)$ is measurable. ☐

Proposition 28

$A \subseteq \mathbb{R}$ measurable, $f : A \to \mathbb{R}$, the following are equivalent,

1. f is measurable
2. $\forall a \in \mathbb{R},\ f^{-1}(a, \infty)$ is measurable
3. $\forall a < b,\ f^{-1}(a, b)$ measurable

Proof.

- $1) \implies 2)$, trivial

- $2) \implies 3)$, let $b \in \mathbb{R}$ such that $f^{-1}(b, \infty)$ is measurable, then $\mathbb{R} \setminus f^{-1}(b, \infty) = f^{-1}(\mathbb{R} \setminus (b, \infty) = f^{-1}((-\infty, b])$ is measurable as well.
  We see that $(-\infty, b) = \bigcup_{n=1}^{\infty}(-\infty, b - \frac{1}{n}]$ and so

  $$f^{-1}(-\infty, b) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, b - \frac{1}{n}])$$

  so it's measurable.
  Finally, for $a < b$,

  $$(a, b) = (a, \infty) \cap (-\infty, b)$$

  so

  $$f^{-1}((a, b)) = f^{-1}((a, \infty) \cap (-\infty, b)) = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b))$$

  so it's measurable.

- $3) \implies 1)$ Trivial. Any open set is a countable union of intervals.

$\square$

1.13 Properties of Measurable Function

Proposition 29

$A \subseteq \mathbb{R}$ measurable, $f, g : A \to \mathbb{R}$ measurable.

1. $\forall a, b \in \mathbb{R},\ af + bg$ is measurable
2. The function fg is measurable.

Proof.

1. Let $a \in \mathbb{R}$. For $\alpha \in \mathbb{R}$, $(af)^{-1}(\alpha, \infty) = \{x \in A : af(x) > \alpha\}$

   (a) if $a > 0$,

   $$(af)^{-1}(\alpha, \infty) = \{x \in A : f(x) > \alpha/a\} = f^{-1}(\alpha/a, \infty) \implies \text{measurable}$$

   (b) $a < 0$,

   $$(af)^{-1}(\alpha, \infty) = f^{-1}(-\infty, \alpha/a) \implies \text{measurable}$$

   (c) $a = 0$,

   $$af \text{ constant} \implies \text{continuous} \implies \text{measurable}$$

   We now show that $f + g$ measurable. For $\alpha \in \mathbb{R}$,

   $$\begin{aligned}(f+g)^{-1}(\alpha, \infty) &= \{x \in A : f(x) + g(x) > \alpha\} \\ &= \{x \in A : f(x) > \alpha - g(x)\} \\ &= \{x \in A : \exists q \in \mathbb{Q}, f(x) > q > \alpha - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} (\{x \in A : f(X) > q\} \cap \{x \in A : g(x) > \alpha - q\}) \\ &= \bigcup_{q \in \mathbb{Q}} f^{-1}(q, \infty) \cap g^{-1}(\alpha - q, \infty) \implies \text{measurable}\end{aligned}$$

   so $f + g$ is measurable.

2. By the quiz, $|f|$ is measurable. For $\alpha \in \mathbb{R}$,

   $$\begin{aligned}&(f^2)^{-1}(\alpha, \infty) \\ =&\{x \in A : f(x)^2 > \alpha\} \\ =&\begin{cases} A, & \alpha < 0 \\ \{x \in A : |f(x)| > \sqrt{\alpha}\}, & \alpha \geqslant 0 \end{cases} \\ =&\begin{cases} A, & \alpha < 0 \\ |f|^{-1}(\sqrt{\alpha}, \infty), & \alpha \geqslant 0 \end{cases}\end{aligned}$$

is measurable, so f^2 is measurable.
Since $(f+g)^2$ is also measurable, and

$$2fg = (f+g)^2 - f^2 - g^2$$

so $2fg$ is measurable. By 1),

$\square$

Example 5

$\psi : [0,1] \to \mathbb{R}$, $\psi(x) = x + \varphi(x)$. There exists $A \subseteq [0,1]$ such that A is measurable but $\psi(A)$ is not measurable. Extend $\psi : \mathbb{R} \to \mathbb{R}$ continuously to a strictly increasing surjective function such that ψ^{-1} is continuous. Consider $\chi_A \circ \psi^{-1}$ where both χ_A and ψ^{-1} are measurable. Then,

$$\begin{aligned} &(\chi_A \circ \psi^{-1})^{-1}\left(\frac{1}{2}, \frac{3}{2}\right) \\ =&\psi(\chi_A^{-1}(1/2, 3/2)) \\ =&\psi(A) \text{ NOT measurable} \end{aligned}$$

Proposition 30

$A \subseteq \mathbb{R}$ measurable. If $g : A \to \mathbb{R}$ is measurable and $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f \circ g$ is measurable.

Proof. Let $U \subseteq \mathbb{R}$ open, then

$$(f \circ g)^{-1}(U) = g^{-1}(\underbrace{f^{-1}(U)}_{\text{open}})$$

which is always measurable by g being measurable. $\square$

1.14 More Properties for Measurable Functions

Definition 7

$A \subseteq \mathbb{R}$, we say a property $P(x)$ ($x \in A$) is true almost everywhere if

$$m(\{x \in A : P(x) \text{ false}\}) = 0$$

Proposition 31

$f : A \to \mathbb{R}$ measurable. If $g : A \to \mathbb{R}$ is a function and $f = g$ a.e., then g is measurable.

Proof. $B := \{x \in A : f(x) \neq g(x)\}$, and $m(B) = 0$. Let $\alpha \in \mathbb{R}$, then

$$\begin{aligned} g^{-1}(\alpha, \infty) &= \{x \in A : g(x) > \alpha\} \\ &= \{x \in A \setminus B : g(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= \{x \in A \setminus B : f(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= (\underbrace{f^{-1}(\alpha, \infty)}_{\text{measurable}} \cap \underbrace{A \setminus B}_{A, B \text{measurable, so it's measurable}}) \cup \underbrace{\{x \in B : g(x) > \alpha\}}_{\subseteq B, \text{so measure zero, measurable}} \end{aligned}$$

Hence, $g^{-1}(a, \infty)$ is measurable, so g is measurable. $\square$

Proposition 32

A is measurable, and $B \subseteq A$ is measurable. A function $f : A \to \mathbb{R}$ is measurable if and only if $f|_B$ and $f|_{A \setminus B}$ are measurable.

Proof.

- $\implies$ Suppose $f : A \to \mathbb{R}$ is measurable. Let $\alpha \in \mathbb{R}$, then,

  $$(f|_B)^{-1}(\alpha, \infty) = \{x \in B : f(x) > \alpha\} = f^{-1}(\alpha, \infty) \cap B \implies \text{ measurable}$$

  so $f|_B$ is measurable, the proof for $f|_{A \setminus B}$ is identical.

- $\impliedby$ Suppose $f|_B$ and $f|_{A \setminus B}$ are measurable. For $\alpha \in \mathbb{R}$,

  $$\begin{aligned} f^{-1}(\alpha, \infty) &= \{x \in A : f(x) > \alpha\} \\ &= \{x \in B : f(x) > \alpha\} \cup \{x \in A \setminus B : f(x) > \alpha\} \\ &= (f|_B)^{-1}(\alpha, \infty) \cup (f|_{A \setminus B})^{-1}(\alpha, \infty) \end{aligned}$$

  is measurable, so f is measurable.

$\square$

Proposition 33

(f_n) measurable, $A \to \mathbb{R}$. If $f_n \to f$ pointwise a.e. then f is measurable.

Proof. Let $B = \{x \in A : f_n(x) \not\to f(x)\}$ so that $m(B) = 0$.
For $\alpha \in \mathbb{R}$,

$$(f|_B)^{-1}(\alpha, \infty) = \underbrace{f^{-1}(\alpha, \infty) \cap B}_{\text{measure zero}} \text{ is measurable}$$

It suffices to show that $f|_{A\setminus B}$ is measurable. By replacing f by $f|_{A\setminus B}$, we may assume $f_n \to f$ pointwise. Let $\alpha \in \mathbb{R}$, since $f_n \to f$ pointwise, we set that for $x \in A$,

$$f(x) > \alpha \iff \exists n, N \in \mathbb{N}, \forall i \in \mathbb{N}, f_i(x) > \alpha + \frac{1}{n} (\text{ to avoid } f_n \to \alpha)$$

We then see that

$$\begin{aligned} & f^{-1}(\alpha, \infty) \\ = & \bigcup_{n \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \underbrace{f_i^{-1}(\alpha + \frac{1}{n}, \infty)}_{\text{measurable}} \end{aligned}$$

is measurable, which implies that f is measurable. $\square$

1.15 Simple Approximation

Definition 8

A function $\varphi : A \to \mathbb{R}$ is called simple if

1. φ is measurable
2. $\varphi(A)$ is finite

Remark. [Conical Representation]

$$\varphi : A \to \mathbb{R} \text{ is simple}$$

and

$$\varphi(A) = \{\underbrace{c_1, c_2, \ldots, c_k}_{\text{distinct}}\}$$

then

$$A_i = \varphi^{-1}(\{c_i\}) \text{ measurable}$$
$$A = \dot{\bigcup}_{i=1}^{k} A_i$$
$$\varphi = \sum_{i=1}^{k} c_i \chi_{A_i}$$

Lemma 34

$f : A \to \mathbb{R}$ measurable and bounded. $\forall \varepsilon > 0$, there exists simple function, $\varphi_\varepsilon, \psi_\varepsilon : A \to \mathbb{R}$ such that $\forall x \in A$,

1. $\varphi_\varepsilon \leqslant f \leqslant \psi_\varepsilon$ and
2. $0 \leqslant \psi_\varepsilon - \varphi_\varepsilon < \varepsilon$

Proof.

$$f(A) \subseteq [a, b]$$

Given $\varepsilon > 0$,

$$\begin{aligned}
&a = y_0 < y_1 < y_2 \ldots < y_n = b \\
&y_{i+1} - y_i < \varepsilon \\
&\underbrace{I_k}_{\text{Borel}} = [y_{k-1}, y_k),\ A_k = f^{-1}(I_k) \implies \text{ measurable} \\
&\varphi_\varepsilon : A \to \mathbb{R}, \psi_\varepsilon : A \to \mathbb{R} \\
&\varphi_\varepsilon = \sum_{k=1}^{n} y_{k-1}\chi_{A_k} \\
&\psi_\varepsilon = \sum_{k=1}^{n} y_k\chi_{A_k}
\end{aligned}$$

Let $x \in A$. Since $f(x) \in [a, b]$, $\exists k \in \{1, \ldots, n\}$ such that $f(x) \in I_k$ i.e. $y_{k-1} \leqslant f(x) \leqslant y_k$, $x \in A_k$. Moreover,

$$\varphi_\varepsilon(x) = y_{k-1} \leqslant f(x) \leqslant y_k = \psi_\varepsilon(x)$$

and so

$$\varphi_\varepsilon \leqslant f \leqslant \psi_\varepsilon$$

For the same x,

$$0 \leqslant \psi_\varepsilon(x) - \varphi_\varepsilon(x) = y_k - y_{k-1} < \varepsilon$$

□

Theorem 35: Simple Approximation

$A \subseteq \mathbb{R}$ is measurable. A function $f : A \to \mathbb{R}$ is measurable if and only if there is a sequence (φ_n) of simple functions on A such that

1. $\varphi_n \to f$ pointwise
2. $\forall n,\ |\varphi_n| \leqslant |f|$

Proof.

- $\Longleftarrow$ Simple functions are measurable and pointwise limit of measurable functions is also measurable
- $\Longrightarrow$ Suppose $f : A \to \mathbb{R}$ is measurable,
  1. $f \geqslant 0$
     For $n \in \mathbb{N}$, define
     $$A_n = \{x \in A : f(x) \leqslant n\}$$
     such that A_n is measurable and $f|_{A_n}$ is measurable and bounded.
     By the lemma, there exists simple functions φ_n and ψ_n such that
     $$0 \leqslant \varphi_n \leqslant f \leqslant \psi_n \text{ on } A_n \text{ and } 0 \leqslant \psi_n - \varphi_n < \frac{1}{n}$$

Fix $n \in \mathbb{N}$, extend $\varphi_n : A \to \mathbb{R}$ by setting $\varphi_n(x) = n$ if $x \notin A_n$, so $0 \leqslant \varphi_n \leqslant f$
For each $n \in \mathbb{N}$, $\varphi_n : A \to \mathbb{R}$ is simple (it's just a simple function with one more value on a disjoint set).
<u>Claim:</u> $\varphi_n \to f$ pointwise
Let $x \in A$ and let $N \in \mathbb{N}$ such that $f(x) \leqslant N$ (i.e. $x \in A_N$). For $n \geqslant N,\ x \in A_n$ and so

$$0 \leqslant f(x) - \varphi_n(x) \leqslant \psi_n(x) - \varphi_n(x) < \frac{1}{n}$$

2. $f : A \to \mathbb{R}$ is measurable. And $B = \{x \in A : f(x) \geqslant 0\}$ and $C = \{x \in A : f(x) < 0\}$ are both measurable.
Define $g, h : A \to \mathbb{R}$,

$$g = \chi_B f,\ h = -\chi_B f$$

so that g, h measurable and non-negative.
By Case 1, there exists a sequence (φ_n), (ψ_n) of simple functions such that $\varphi_n \to g$ pointwise, $\psi_n \to h$ pointwsie, $0 \leqslant \varphi_n \leqslant g,\ 0 \leqslant \psi_n \leqslant h$. Then

$$\underbrace{\varphi_n - \psi_n}_{\text{simple}} \to g - h = f \text{ pointwise}$$

and

$$|\varphi_n - \psi_n| \leqslant |\psi_n| + |\varphi_n| = \varphi_n + \psi_n \leqslant g + h = |f|$$

$\square$

1.16 Littlewood's Principle

Up to certain finiteness conditions

1. Measurable sets are "almost" finite, disjoint unions of bounded open intervals.
2. Measurable functions are "almost" continuous.
3. Pointwise limits of measurable functions are "almost" uniform limits

Theorem 36: [Littlewood 1]

A be measurable set, $m(A) < \infty$. $\forall \varepsilon > 0$, there exists finitely many open, bounded, disjoint intervals $I_1, I_2, \ldots, I_n$ such that $m(A \triangle U) < \varepsilon$, where $U = I_1 \cup I_2 \cup \ldots \cup I_n$. Note: $m(A \triangle U) = m(A \setminus U) + m(U \setminus A)$.

Proof. Let $\varepsilon > 0$ be given. We may find an open set U and $A \subseteq U$ and

$$m(U \setminus A) < \frac{\varepsilon}{2}$$

By PMATH351, there exists open, bounded, disjoint intervals $I_i (i \in \mathbb{N})$ such that

$$U = \dot{\bigcup}_{i=1}^{\infty} I_i$$

Note that,

$$\sum_{i=1}^{\infty} l(I_i) = m(U) = m(U \setminus A) + m(A) < \infty$$

In particular, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} l(I_i) = \frac{\varepsilon}{2}$$

Take $V = I_1 \cup \ldots \cup I_N$, we see that

$$\begin{aligned} m(A \setminus V) &\leqslant m(U \setminus V) \\ &= m\left(\bigcup_{N+1}^{\infty} I_i\right) \\ &= \sum_{N+1}^{\infty} l(I_i) < \frac{\varepsilon}{2} \end{aligned}$$

and

$$m(V \setminus A) \leqslant m(U \setminus A) < \frac{\varepsilon}{2}$$

□

Lemma 37

Let A be measurable and $m(A) < \infty$, (f_n) be measurable, $A \to \mathbb{R}$. Assume $f : A \to \mathbb{R}$ such that $f_n \to f$ pointwise. $\forall \alpha, \beta > 0$, there exists a measurable subset $B \subseteq A$ and $N \in \mathbb{N}$ such that

1. $|f_n(x) - f(x)| < \alpha, \forall\, x \in B, n \geqslant N$
2. $m(A \setminus B) < \beta$

Proof. Let $\alpha, \beta > 0$ be given. For $n \in \mathbb{N}$, define

$$\begin{aligned} A_n &= \{x \in A : |f_k(x) - f(x)| < \alpha, \forall k \geqslant n\} \\ &= \bigcap_{k=n}^{\infty} \underbrace{|f_k - f|^{-1}(-\infty, \alpha)}_{\text{measurable}} \end{aligned}$$

So every A_n is measurable. Since $f_n \to f$ pointwise,

$$A = \bigcup_{n=1}^{\infty} A_n$$

Since (A_n) is ascending, by continuity of measure,

$$m(A) = \lim_{n\to\infty} m(A_n) < \infty$$

we may find $N \in \mathbb{N}$ such that $\forall n \geqslant N$,

$$m(A) - m(A_n) < \beta$$

Pick $B = A_N$ we get what's required. $\square$

Theorem 38: Littlewood 3, Egoroff's Theorem

A is measurable, $m(A) < \infty$, (f_n) is measurable, $A \to \mathbb{R}$, $f_n \to f$ pointwise. $\forall \varepsilon > 0$, there exists a closed set $C \subseteq A$ such that

1. $f_n \to f$ uniformly on C
2. $m(A \setminus C) < \varepsilon$

Proof. Let $\varepsilon > 0$ be given. By the lemma, for every $n \in \mathbb{N}$, there exists a measurable set $A_n \subseteq A$ and $N(n) \in \mathbb{N}$ such that

1. $\forall x \in A_n$ and $k \geqslant N(n)$,

$$|f_k(x) - f(x)| < \frac{1}{n}$$

2. $m(A \setminus A_n) < \frac{\varepsilon}{2^{n+1}}$

Take $B = \bigcap_{n=1}^{\infty} A_n$ (measurable). For $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon,\ k \geqslant N(n)$, and $x \in B$,

$$|f_k(x) - f(x)| < \frac{1}{n} < \varepsilon$$

so $f_n \to f$ uniformly on B. Moreover,

$$m(A \setminus B) = m(A \setminus \cap A_n) = m(\cup(A \setminus A_n)) \leqslant \sum m(A \setminus A_n) < \sum \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}$$

By A1, there exists a closed set C such that $C \subseteq B$ and $m(B \setminus C) < \frac{\varepsilon}{2}$, so

1. Since $C \subseteq B$, $f_k \to f$ uniformly on C
2. $m(A \setminus C) = m(A \setminus B) + m(B \setminus C) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

□

Warning:
$f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \frac{x}{n}$ and $f_n \to 0$ pointwise. But $f_n \not\to 0$ uniformly on any measurable set $B \subseteq \mathbb{R}$ such that $m(\mathbb{R} \setminus B) < 1$

Proof. Suppose such B exists. Since B measurable, $B \subseteq \mathbb{R}$, we know

$$m(\mathbb{R} \setminus B) = m(\mathbb{R}) - m(B) < 1 \implies m(B) = \infty$$

That is, B has to be unbounded.
Since $f_n \to 0$ uniformly on B, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, s/t\ \forall k \geqslant N, \forall x \in B$,

$$|0 - f_k(x)| < \varepsilon \implies \left|\frac{x}{k}\right| < \varepsilon$$

However, since B is unbounede, we can always find $x \in B$ such that $|x| = (\varepsilon + 1)|k|$, so $|x/k| = \varepsilon + 1 > \varepsilon$.
That is, no matter how big the N is, I can always find points where the uniformly convergence condition fails. Contradiction! So no such B exists. □

Lemma 39

$f : A \to \mathbb{R}$ simple. $\forall \varepsilon > 0$, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a closed $C \subseteq A$ such that

1. $f = g$ on C
2. $m(A \setminus C) < \varepsilon$

Proof. $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, conical representation. $A_i = \{x \in A : f(x) = a_i\}$ is measurable. By A1, $C_i \subseteq A_i$ closed,

$$m(A_i \setminus C_i) < \frac{\varepsilon}{n}$$

AND

$$A = \dot{\bigcup}_{i=1}^{n} A_i,\ C := \dot{\bigcup}_{i=1}^{n} C_i \text{ closed}$$

1. $\forall x \in C_i, f(x) = a_i$. By A1, f is continuous on $C \implies$ we then extend $f|_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$

2. $m(A \setminus C) = m(\cup_{i=1}^n A_i \setminus C_i) = \sum_{i=1}^n m(A_i \setminus C_i) < \varepsilon$

$\square$

Theorem 40: Littlewood 2, Lusin Theorem

$f : A \to \mathbb{R}$ is measurable. $\forall \varepsilon > 0$, there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed set $C \subseteq A$ such that

1. $f = g$ on C and

2. $m(A \setminus C) < \varepsilon$

Proof. Let $\varepsilon > 0$ given.

1. $m(A) < \infty$
   Let $f : A \to \mathbb{R}$ be measurable. By the Simple Approximation Theorem, there exists (f_n) simple such that $f_n \to f$ pointwise. By the lemma, there exists continuous $g_n : \mathbb{R} \to \mathbb{R}$ and closed $C_n \subseteq A$ such that

   (a) $f_n = g_n$ on C_n
   (b) $m(A \setminus C_n) < \frac{\varepsilon}{2^{n+1}}$

   By Egoroff, there exists a closed set $C_0 \subseteq A$ such that $f_n \to f$ uniformly on C_0 and $m(A \setminus C_0) < \frac{\varepsilon}{2}$.

   Let $C = \bigcap_{i=0}^\infty C_i$

   (a) $g_n = f_n \to f$ uniformly on $C \subseteq C_0$, so f is continuous on C. By A1, extend $f|_C$ to a continuouse function $g : \mathbb{R} \to \mathbb{R}$.

   (b)
   $$\begin{aligned} m(A \setminus C) &= m(A \setminus \cap_{i=0}^\infty C_i) = m(\cup_{i=0}^\infty (A \setminus C_i)) \\ &\leqslant \sum_{i=0}^\infty m(A \setminus C_i) = m(A \setminus C_0) + \sum_{i=1}^\infty m(A \setminus C_i) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

2. $m(A) = \infty$
   For $n \in \mathbb{N}$,
   $$A_n = \{a \in A : |a| \in [n-1, n)\}$$
   such that
   $$A = \dot{\bigcup}_{n=1}^\infty A_n$$
   By case 1, there exists continuous functions $g_n : \mathbb{R} \to \mathbb{R}$ and closed $C_n \subseteq A_n$ such that

(a) $f = g_n$ on C_n

(b) $m(A_n \setminus C_n) < \frac{\varepsilon}{2^n}$

Consider $C = \dot{\bigcup}_{n=1}^{\infty} C_n$, and C is closed.

(a) $m(A \setminus C) = m(\dot{\cup}(A_n \setminus C_n)) = \sum m(A_n \setminus C_n) < \varepsilon$

(b) $g : C \to \mathbb{R}$. Let $x \in C$ such that $x \in C_n$ for one $n \in \mathbb{N}$. Define $g(x) = g_n(x) = f(x)$. By A1, extend g on $\mathbb{R}$.

$\square$

2 Integration

2.1 Integration

1. Simple functions
$$\varphi : A \to \mathbb{R},\ m(A) < \infty$$
2. $f : A \to \mathbb{R}$, bounded measure, $m(A) < \infty$,
$$\varphi_\varepsilon \leqslant f \leqslant \psi_\varepsilon$$
3. $f : A \to \mathbb{R}$ measurable, $f \geqslant 0$,
$$\sup\left\{\int_A h : h \in (2), 0 \leqslant h \leqslant f\right\}$$
4. $f : A \to \mathbb{R}$ measurable,
$$f^+ = \max\{f, 0\}$$
$$f^- = \max\{-f, 0\}$$

Step 1: $\varphi : A \to \mathbb{R}$ simple, $m(A) < \infty$

Definition 9

$m(A) < \infty$, $\varphi : A \to \mathbb{R}$ simple. Conical Rep.:$\varphi = \sum_{i=1}^n a_i \chi_{A_i}$. The (Lebesgue) Integral of φ over A is

$$\int_A \varphi = \sum_{i=1}^n a_i m(A_i)$$

Lemma 41

$m(A) < \infty$ (A measurable). If $B_1, B_2, \ldots, B_n \subseteq A$ are measurable and disjoint and $\varphi : A \to \mathbb{R}$ defined by

$$\varphi = \sum_{i=1}^n b_i \chi_{B_i}$$

then

$$\int_A \varphi = \sum_{i=1}^n b_i m(B_i)$$

Proof. For $n = 2$,
If $b_1 \neq b_2$, then $\varphi = b_1\chi_{B_1} + b_2\chi_{B_2}$ is the conical representation.
If $b_1 = b_2$, then

$$b_1\chi_{B_1} + b_1\chi_{B_2} = b_1(\chi_{B_1} + \chi_{B_2}) = \underbrace{b_1\chi_{B_1\cup B_2}}_{\text{conical rep.}}$$

so

$$\begin{aligned}\int_A \varphi &= b_1 m(B_1 \dot{\cup} B_2) \\ &= b_1(m(B_1) + m(B_2)) \\ &= b_1 m(B_1) + b_2 m(B_2)\end{aligned}$$

Then simple dicuss other cases. □

Proposition 42

$\varphi, \psi : A \to \mathbb{R}$ simple, $m(A) < \infty$. For all $\alpha, \beta \in \mathbb{R}$,

$$\int_A (\alpha\varphi + \beta\psi) = \alpha \int_A \varphi + \beta \int_A \psi$$

Proof.

$$\varphi(A) = \{a_1, a_2, \ldots, a_n\}$$
$$\psi(A) = \{b_1, b_2, \ldots, b_m\}$$

where the elements are distinct for each set.

Define

$$C_{ij} = \{x \in A : \varphi(x) = a_i, \psi(x) = b_j\} = \varphi^{-1}(\{a_i\}) \cap \psi^{-1}(\{b_j\})$$

which is measurable.

$$\alpha\varphi + \beta\psi = \sum_{i,j} (\alpha a_i + \beta b_j) \chi_{C_{ij}}$$

By the lemma,

$$\begin{aligned}\int_A \alpha\varphi + \beta\psi &= \sum_{i,j} (\alpha a_i + \beta b_j) m(C_{ij}) \\ &= \sum_{i,j} \alpha a_i m(C_{ij}) + \sum_{i,j} \beta b_j m(C_{ij}) \\ &= \sum_i \alpha a_i \sum_j m(C_{ij}) + \sum_j \beta b_j \sum_i m(C_{ij}) \\ &= \sum_i \alpha a_i m(\{x \in A : \varphi(x) = a_i\}) + \sum_j \beta b_j m(\{x \in A : \varphi(x) = a_i\}) \\ &= \alpha \int_A \varphi + \beta \int_A \psi\end{aligned}$$

□

Proposition 43

$\varphi, \psi : A \to \mathbb{R}$ simple, $m(A) < \infty$. If $\varphi \leqslant \psi$, then

$$\int_A \varphi \leqslant \int_A \psi$$

Proof.

$$\int_A \psi - \int_A \varphi = \int_A \underbrace{(\psi - \varphi)}_{\geqslant 0} \geqslant 0$$

$\square$

Step2: $f : A \to \mathbb{R}$ bounded, measurable $m(A) < \infty$

Definition 10

$f : A \to \mathbb{R}$ be bounded, measurable and $m(A) < \infty$. Then

- Lower Lebesgue Integral:

$$\underline{\int_A} f = \sup \left\{ \int_A \varphi : \varphi \leqslant f \text{ simple} \right\}$$

- Lower Lebesgue Integral:

$$\overline{\int_A} f = \inf \left\{ \int_A \psi : f \leqslant \psi \text{ simple} \right\}$$

Proposition 44

$m(A) < \infty$, $f : A \to \mathbb{R}$ bounded, measurable. Then

$$\underline{\int_A} f = \overline{\int_A} f$$

Proof. $\forall n \in \mathbb{N}$, there exists simple functions, $\varphi_n, \psi_n : A \to \mathbb{R}$ such that

1. $\varphi_n \leqslant f \leqslant \psi_n$
2. $\psi_n - \varphi_n \leqslant \frac{1}{n}$

We see that

$$\begin{aligned} 0 &\leqslant \overline{\int_A} f - \underline{\int_A} f \\ &\leqslant \int_A \psi_n - \int_A \varphi_n \\ &= \int_A (\psi_n - \varphi_n) \\ &\leqslant \int_A \frac{1}{n} \\ &= \frac{1}{n} m(A) < \infty \\ &\to 0 \end{aligned}$$

□

Definition 11

$m(A) < \infty$, $f : A \to \mathbb{R}$ bounded, measurable, we define the (Lebesgue) <u>integral of f over A</u> by

$$\int_A f := \underline{\int_A} f = \overline{\int_A} f$$

Proposition 45

$f, g : A \to \mathbb{R}$ bounded, measurable, $m(A) < \infty$. For $\alpha, \beta \in \mathbb{R}$,

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

Proof. Scalar multiplication is easy.
Now, have $\varphi_1, \varphi_2, \psi_1, \psi_2$ all simple,

$$\varphi_1 \leqslant f \leqslant \psi_1,\ \varphi_2 \leqslant g \leqslant \psi_2$$

1.
$$\begin{aligned}\int_A f + g &= \overline{\int_A} f + g \\ &\leqslant \int_A \psi_1 + \psi_2 \\ &= \int_A \psi_1 + \int_A \psi_2\end{aligned}$$

so

$$\begin{aligned}\int_A f + g &\leqslant \inf\left\{\int_A \psi_1 + \int_A \psi_2 : f \leqslant \psi_1, g \leqslant \psi_2, \psi_1, \psi_2 \text{ simple}\right\} \\ &= \inf\left\{\int_A \psi_1 : f \leqslant \psi_1 \text{ simple}\right\} + \inf\left\{\int_A \psi_2 : g \leqslant \psi_2 \text{ simple}\right\} \\ &= \int_A f + \int_A g\end{aligned}$$

2.
$$\int_A f + g = \underline{\int_A} f + g \geqslant \int_A \varphi_1 + \int_A \varphi_2$$

so

$$\begin{aligned}\int_A f + g &\geqslant \sup\left\{\int_A \varphi_1 + \int_A \varphi_2 : f \geqslant \varphi_1, g \geqslant \varphi_2, \varphi_1, \varphi_2 \text{ simple}\right\} \\ &= \sup\left\{\int_A \varphi_1 : f \geqslant \varphi_1, \varphi_1 \text{ simple}\right\} + \sup\left\{\int_A \varphi_2 : f \geqslant \varphi_2, \varphi_2 \text{ simple}\right\} \\ &= \int_A f + \int_A g\end{aligned}$$

so

$$\int_A f + g = \int_A f + \int_A g$$

$\square$

Proposition 46

$f, g : A \to \mathbb{R}$ bounded, measurable and $m(A) \leqslant \infty$. If $f \leqslant g$, then $\int_A f \leqslant \int_A g$.

Proof. Since $g - f \geqslant 0$, where 0 is also a simple function, we have

$$\int_A (g - f) = \underline{\int_A} (g - f) \geqslant \int_A 0 = 0 \implies \int_A g \geqslant \int_A f$$

$\square$

2.2 Bounded Convergence Theorem

Proposition 47

$f : A \to \mathbb{R}$ bounded, measurable, $B \subseteq A$ measurable, $m(A) < \infty$, then

$$\int_B f = \int_A f\chi_B$$

Proof.

1. $f = \chi_C,\ C \subseteq A$ measurable.

$$\begin{aligned}\int_A \chi_C\chi_B &= \int_A \chi_{B\cap C}\\ &= 1 * m(B\cap C)\\ &= \int_B \chi_{C|_B}\end{aligned}$$

2. f is simple, $f = \sum_{i=1}^n a_i\chi_{A_i}$,

$$\int_A f\chi_B = \sum a_i \int_A \chi_{A_i}\chi_B = \sum a_i \int_B \chi_{A_i} = \int_B (\sum a_i\chi_{A_i|_B}) = \int_B f$$

3. $f : A \to \mathbb{R}$ be bounded and measurable.
   First we take $f \leqslant \psi$, simple, then

$$\int_A f\chi_B \leqslant \int_A \psi\chi_B = \int_B \psi$$

By taking the $\inf$ over all such ψ, we have that

$$\int_A f\chi_B \leqslant \overline{\int_A} f = \int_B f$$

Similarly, taking $\varphi \leqslant f$, φ simple, we obtain,

$$\underline{\int_B} f = \int_B f \leqslant \int_A f\chi_B$$

so we have

$$\int_A f\chi_B = \int_B f$$

$\square$

Proposition 48

$f : A \to \mathbb{R}$ be bounded, measurable, $m(A) < \infty$. If $B, C \subseteq A$ are measurable and disjoint, then

$$\int_{B\cup C} f = \int_B f + \int_C f$$

Proof.

$$\begin{aligned}\int_{B\cup C} f &= \int_A f\chi_{B\cup C} \\ &= \int_A f(\chi_B + \chi_C) \\ &= \int_A f\chi_B + \int_A f\chi_C \\ &= \int_B f + \int_C f\end{aligned}$$

□

Proposition 49

$f : A \to \mathbb{R}$ be bounded, measurable, $m(A) < \infty$, then $\left|\int_A f\right| \leqslant \int_A |f|$.

Proof.

$$\begin{aligned}-|f| \leqslant f &\leqslant |f| \\ -\textstyle\int_A |f| \leqslant \int_A |f| &\leqslant \textstyle\int_A |f|\end{aligned}$$

□

Proposition 50

(f_n) is bounded, measurable, $A :\to \mathbb{R}$, $m(A) < \infty$. If $f_n \to f$ uniformly, then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Let $\varepsilon > 0$ be given, let $N \in \mathbb{N}$ such that

$$|f_n - f| \leqslant \frac{\varepsilon}{m(A) + 1}$$

then, for $n \geqslant N$

$$\begin{aligned}
&\left|\int_A f_n - \int_A f\right| \\
=&\left|\int_A (f_n - f)\right| \\
\leqslant& \int_A |f_n - f| \\
\leqslant& m(A) * \frac{\varepsilon}{m(A)+1} \\
<& \varepsilon
\end{aligned}$$

□

Example 6

$f_n : [0,1] \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 0, & 0 \leqslant x < \frac{1}{n} \\ n, & \frac{1}{n} \leqslant x < \frac{2}{n} \\ 0, & \frac{2}{n} \leqslant x \end{cases}$$

then $f_n \to 0$ pointwisely, but

$$\int_{[0,1]} f_n = 1, \int_{[0,1]} 0 = 0$$

Theorem 51: [BCT]

$(f_n) : A \to \mathbb{R}$ measurable, $m(A) < \infty$. If there exists $M > 0$ such that $|f_n| \leqslant M$ for all n and $f_n \to f$ pointwise then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Let $\varepsilon > 0$ be given. By Egoroff's theorem, there exists measurable $B \subseteq A$ and $N \in \mathbb{N}$ such that for $n \geqslant N$,

1. $|f_n - f| < \frac{\varepsilon}{2(m(B)+1)}$ on B
2. $m(A \setminus B) < \frac{\varepsilon}{4M}$

$\forall n \geqslant N,$

$$\begin{aligned}
\left|\int_A f_n - \int_A f\right| &\leqslant \int_A |f_n - f| \\
&= \int_B |f_n - f| + \int_{A \setminus B} |f_n - f| \\
&\leqslant \int_B |f_n - f| + \int_{A \setminus B} (|f_n| + |f|) \\
&\leqslant \int_B |f_n - f| + 2M * m(A \setminus B) \\
&=\leqslant m(B) \frac{\varepsilon}{2(M(B)+1)} + 2M \frac{\varepsilon}{4M} \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

$\square$

Definition 12

$f : A \to \mathbb{R}$ measurable

1. We say f has finite support if
$$A_0 := \{x \in A : f(x) \neq 0\}$$
has finite measure.
2. We say f is a BF function. If f is bounded and has finite support.
3. If $f : A \to \mathbb{R}$ is BF, then
$$\int_A f := \int_{A_0} f$$

Definition 13

$f : A \to \mathbb{R}$ measurable, $f \geqslant 0$,
$$\int_A f = \sup \left\{ \int_A h : 0 \leqslant h \leqslant f, \text{ BF} \right\}$$

Proposition 52

$f, g : A \to \mathbb{R}$ measurable, $f, g \geqslant 0$

1. $\forall \alpha, \beta \in \mathbb{R}$,
$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$
2. If $f \leqslant g$, then $\int_A f \leqslant \int_A g$
3. If $B, C \subseteq A$ are measurable and $B \cap C = \emptyset$ then
$$\int_{B \cup C} f = \int_B f + \int_C f$$

Theorem 53: [Chebychev's Inequality]

$f : A \to \mathbb{R}$ measurable, non-negative; $\forall \varepsilon > 0$,
$$m\left(\{x \in A : f(x) \geqslant \varepsilon\}\right) \leqslant \frac{1}{\varepsilon} \int_A f$$

Proof. Let $\varepsilon > 0$ given and let
$$A_\varepsilon = \{x \in A : f(x) \geqslant \varepsilon\}$$

1. $m(A_\varepsilon) < \infty$

$$\underbrace{\varphi}_{\text{BF}} = \varepsilon\chi_{A_\varepsilon} \leqslant f$$

so

$$\varepsilon m(A_\varepsilon) = \int_A \varphi \leqslant \int_A f$$

2. $m(A_\varepsilon) = \infty$ For $n \in \mathbb{N}$, $A_{\varepsilon,n} := A_\varepsilon \cap [-n, n]$. By the continuity of measure,

$$\infty = m(A_\varepsilon) = \lim_{n\to\infty} m(A_{\varepsilon,n})$$

For $n \in \mathbb{N}$, $\varphi_n := \varepsilon\chi_{\varepsilon,n}$(BF), we see that $\varphi_n \leqslant f$.
Therefore,

$$\begin{aligned}\infty =& m(A_\varepsilon)\\ =& \lim_{n\to\infty} m(A_{\varepsilon,n})\\ =& \lim_{n\to\infty} \frac{1}{\varepsilon}\int_A \varphi_n\\ \leqslant& \frac{1}{\varepsilon}\int_A f\end{aligned}$$

□

Proposition 54

$f : A \to \mathbb{R}$ measurable, $f \geqslant 0$

$$\int_A f = 0 \iff f = 0 \text{ a.e.}$$

Proof.

- ($\implies$) Suppose $\int_A(f) = 0$,

$$\begin{aligned}& m\left(\{x \in A : f(x) \neq 0\}\right)\\ \leqslant& \sum m\left(\left\{x \in A : f(x) \geqslant \frac{1}{n}\right\}\right)\\ \underbrace{\leqslant}_{\text{Chebychev}}& \sum n \int_A f = 0\end{aligned}$$

- $\impliedby$ Suppose $B = \{x \in A : f(x) \neq 0\}$ has measure 0.

$$\begin{aligned}\int_A f =& \int_B f + \int_{A\setminus B} \underbrace{f}_{=0}\\ =& \int_B f + 0\\ =& 0\end{aligned}$$

$\int_B f = 0$ because for any h BF and $0 \leqslant h \leqslant f$, there is a $M_h \geqslant 0$ such that $h \leqslant M_h$, then

$$\int_B 0 \leqslant \int_B h \leqslant \int_B M_h = \int_B M_h \chi_B = M_h m(B) = M_h * 0 = 0$$

so $\int_B h$ is always zero, hence

$$\int_B f = \sup \left\{ \int_B h : 0 \leqslant h \leqslant f,\ h \text{ BF} \right\} = 0$$

□

2.3 Fatou's Lemma and MCT

Theorem 55: Fatou's Lemma

(f_n) measurable, non-negative, $A \to \mathbb{R}$. If $f_n \to f$ pointwise then

$$\int_A f \leqslant \liminf \int_A f_n$$

Proof. Let $0 \leqslant h \leqslant f$ be a BF function. Say $A_0 = \{x \in A : h(x) \neq 0\}$. It suffices to show

$$\int_A h \leqslant \liminf \int_A f_n$$

Since h is BF, $m(A_0) < \infty$. For each $n \in \mathbb{N}$, let

$$h_n = \min\{h, f_n\} \text{ (meas.)}$$

Note:

1. $0 \leqslant h_n \leqslant h \leqslant M$, for some $M > 0$, $\forall n \in \mathbb{N}$
2. For $x \in A_0$ and $n \in \mathbb{N}$,
   (a) $h_n(x) = h(x)$ or
   (b) $h_n(x) = f_n(x) \leqslant h(x)$ and
   $$0 \leqslant h(x) - h_n(x) = h(x) - f_n(x) \leqslant f(x) - f_n(x) \to 0$$
   so $h_n(x) \to h$ on A_0

By BCT,

$$\lim_{n\to\infty} \int_{A_0} h_n = \int_{A_0} h \implies \lim_{n\to\infty} \int_A h_n = \int_A h$$

Since $h_n \leqslant f_n$ on A,

$$\int_A = \lim_{n\to\infty} \int_A h_n = \liminf_{n\to\infty} \int_A h_n \leqslant \liminf_{n\to\infty} \int_A f_n$$

□

Example 7

$$\begin{aligned}
&A = (0,1] \\
&f_n = n\chi(0,1/n) \\
&f_n \to 0 \text{ pointwise} \\
&\int_A 0 = 0 \\
&\int_A f_n = n \cdot m(0,1/n) = 1 \\
&\liminf \int_A f_n = 1
\end{aligned}$$

Theorem 56: [MCT]

(f_n) non-negative, measurable, $A \to \mathbb{R}$. If (f_n) is increasing and $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof.

$$\begin{aligned}
\int_A f &\leqslant \liminf \int_A f_n \text{ by Fatou's Lemma} \\
&\leqslant \limsup \int_A f_n \\
&\leqslant \int_A f \text{ by } f_n \nearrow \text{and converge to } f
\end{aligned}$$

so $\lim_{n\to\infty} \int_A f_n = \liminf \int_A f_n = \limsup \int_A f_n$ □

Remark.

1. If $\varphi : A \to \mathbb{R}$ is simple and $m(A) < \infty$, then

$$\int_A \varphi < \infty$$

2. If $f : A \to \mathbb{R}$ is bounded, measurable and $m(A) < \infty$, then

$$\int_A f < \infty$$

Definition 14

If $f : A \to \mathbb{R}$ is measurable and $f \geqslant 0$, then we say f is integrable if and only if

$$\int_A f < \infty$$

2.4 The General Integral

Definition 15

$f : A \to \mathbb{R}$ measurable,

$$f^+(x) = \max\{f(x), 0\}$$
$$f^-(x) = \max\{-f(x), 0\}$$

Notes:

1. $f^+ + f^- = |f|$
2. $f^+ - f^- = f$
3. f^+, f^- measurable

Proposition 57

$f : A \to \mathbb{R}$ measurable. Then f^+, f^- are integrable if and only if $|f|$ is integrable.

Proof.

- $|f| = f^+ + f^-$

$$\int_A |f| = \underbrace{\int_A f^+}_{<\infty} + \underbrace{\int_A f^-}_{<\infty} < \infty$$

- $$\int_A f^+ \leqslant \int_A |f| < \infty; \ \int_A f^- \leqslant \int_A |f| < \infty$$

□

Definition 16

$f : A \to \mathbb{R}$ measurable. We say f is integrable if and only if $|f|$ is integrable if and only if f^+, f^- are integrable, and define

$$\int_A f = \int_A f^+ - \int_A f^-$$

Proposition 58: [Comparison Test]

$f : A \to \mathbb{R}$ measurable, $g : A \to \mathbb{R}$ non-negative integrable. If $|f| \leqslant g$ then f is integrable and $\left|\int_A f\right| \leqslant \int_A |f|$

Proof.

1. $\underbrace{\int_A |f|}_{<\infty} \leqslant \int_A g < \infty$
2. $|\int_A f| = \left|\int_A f^+ - \int_A f^-\right| \leqslant |\int_A f^+| + |\int_A f^-| = \int_A f^+ + \int_A f^- = \int_A (f^+ + f^-) = \int_A (f)$

□

Proposition 59

$f, g : A \to \mathbb{R}$ integrable.

1. $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable, and
$$\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$$
2. If $f \leqslant g$, then $\int_A f \leqslant \int_A g$
3. If $B, C \subseteq A$ are measurable with $B \cap C = \emptyset$, then
$$\int_{B \cup C} f = \int_B f + \int_C f$$

Proof.

- Comparison Test
- Results hold for f^+, f^-, g^+, g^-

□

Theorem 60: [Lebesgue Dominated Convergence Theorem]

$f_n : A \to \mathbb{R}$ measurable. $f_n \to f$ pointwise. If there exists a $g : A \to \mathbb{R}$ integrable such that $|f_n| \leqslant g$, $\forall n \in \mathbb{N}$, then f is integrable and $\lim_{n \to \infty} \int_A f_n = \int_A f$

Proof. Since $|f_n| \to |f|$, and so $|f| \leqslant g$.
By comparison test, f is integrable. Next, observe $g - f \geqslant 0$. By Fatou,

1.

$$\begin{aligned}\int_A g - \int_A f = \int_A g - f \\ &\leqslant \liminf \int_A g - f_n \\ &= \int_A g - \limsup \int_A f_n\end{aligned}$$

$$\implies \limsup \int_A f_n \leqslant \int_A f$$

2.

$$\int_A g + \int_A f = \int_A g + f \leqslant \liminf \int_A g + f_n = \int_A g + \liminf \int_A f_n$$

$$\implies \int_A f = \liminf \int_A f_n = \limsup \int_A f_n = \lim \int_A f_n$$

□

2.5 Riemann Integration

Definition 17

$f : [a, b] \to \mathbb{R}$ bounded

1. A partition of $[a, b]$ is a finite set such that

$$P = \{x_0, x_1, \ldots, x_n\} \subseteq \mathbb{R} \text{ and } a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

2. Relative to P, we define the lower Darboux sum:

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

3. Similarly, we define the upper Darboux sum:

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$
$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

Definition 18

$f : [a, b] \to \mathbb{R}$, bounded,

1. Lower Riemann Integral
$$R\underline{\int_a^b} f = \sup \{L(f, P) : P \text{ partition}\}$$
2. Upper Riemann Integral
$$R\overline{\int_a^b} f = \inf \{U(f, P) : P \text{ partition}\}$$
3. We say f is <u>Riemann Integrable</u> if and only if
$$\underbrace{R\underline{\int_a^b} f = R\overline{\int_a^b} f}_{R\int_a^b f}$$

Definition 19

Let $I_1, \ldots, I_n$ be pairwise disjoint intervals such that
$$[a, b] = \dot{\cup}_{i=1}^n I_i$$
A <u>step function</u> is a functions of the form
$$f = \sum_{i=1}^n a_i \chi_{I_i}$$
for some $a_i \in \mathbb{R}$

Remark. $f : [a, b] \to \mathbb{R}$ bounded. $a = x_0 < x_1 < \ldots < x_n = b$. $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$. Then
$$L(f, P) = \sum_{i=1}^n m_i \cdot l(I_i) = R\int_a^b \varphi$$
where $\varphi(x) = mi$ on I_i ($\varphi \leqslant f$).
$$U(f, P) = \sum_{i=1}^n M_i \cdot l(I_i) = R\int_a^b \psi$$
where $\psi(x) = Mi$ on I_i ($f \leqslant \psi$).

Remark. $f : [a, b] \to \mathbb{R}$ bounded,

$$R\underline{\int_a^b} f = \sup\{L(f, P) : P\} = \sup\left\{R\int_a^b \varphi : \varphi \leqslant f \text{ step}\right\}$$
$$R\overline{\int_a^b} f = \inf\{U(f, P) : P\} = \inf\left\{R\int_a^b \psi : f \leqslant \psi \text{ step}\right\}$$

2.5.1 Riemann Integral VS Lebesgue Integral

Definition 20

Let $f : [a, b] \to \mathbb{R}$ bounded. Let $x \in [a, b]$ and $\delta > 0$

1. $m_\delta(x) = \inf\{f(x) : x \in (x - \delta, x + \delta) \cap [a, b]\}$
2. $M_\delta(x) = \sup\{f(x) : x \in (x - \delta, x + \delta) \cap [a, b]\}$
3. Lower Boundary of f,
$$m(x) = \lim_{\delta \to 0} m_\delta(x)$$
4. Upper Boundary of f,
$$M(x) = \lim_{\delta \to 0} M_\delta(x)$$
5. Oscillation of f,
$$\omega(x) = M(x) - m(x)$$

Remark. $f : [a, b] \to \mathbb{R}$ bounded, TFAE

1. f is continuous at $x \in [a, b]$
2. $M(x) = m(x)$
3. $\omega(x) = 0$

Lemma 61

$f : [a, b] \to \mathbb{R}$ bounded,

1. m is measure
2. If $\varphi : [a, b] \to \mathbb{R}$ is a step function with $\varphi \leqslant f$, then $\varphi(x) \leqslant m(x)$ at all points of continuity of φ
3. $R\underline{\int_a^b} f = \int_{[a,b]} m$

Lemma 62

$f : [a, b] \to \mathbb{R}$ bounded,

1. M is measure
2. If $\psi : [a, b] \to \mathbb{R}$ is a step function with $\psi \geqslant f$, then $\psi(x) \geqslant M(x)$ at all points of continuity of ψ
3. $R\overline{\int_a^b} f = \int_{[a,b]} M$

Theorem 63: [Lebesgue]

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f Riemann integrable if and only if f is continuous a.e., in that case,

$$R\int_a^b f = \int_{[a,b]} f$$

Proof.

$$R\underline{\int_a^b} f = \int_{[a,b]} m \leqslant \int_{[a,b]} M = R\overline{\int_a^b} f$$

$$\begin{aligned} & f \text{ Riemann Integrable} \\ \iff & \int_{[a,b]} m = \int_{[a,b]} M \\ \iff & \int_{[a,b]} (\underbrace{M - m}_{\geqslant 0}) = 0 \\ \iff & M = m \text{ a.e.} \\ \iff & \omega = 0 \text{ a.e.} \\ \iff & f \text{ is continuous a.e.} \end{aligned}$$

If f is continuous a.e. $\implies$ f is measurable and

$$R\underline{\int_a^b} f = \int_{[a,b]} m \leqslant \int_{[a,b]} f \leqslant \int_{[a,b]} M = R\overline{\int_a^b} f \implies R\int_a^b f = \int_{[a,b]} f$$

because $M = m$ a.e. $\square$

Example 8

$f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$$

f is discontinuous on $[0, 1] \implies f$ is NOT Riemann Integrable. But $f = 0$ a.e. ans so

$$\int_{[0,1]} f = \int_{[0,1]} 0 = 0$$

Example 9

$\mathbb{Q} \cap [0, 1] = \{q_1, q_2, \ldots\}$, $f_n = \chi_{\{q_1,\ldots,q_n\}}$. $f_n \to f$ pointwise (f in the previous example). f_n is increasing, continuous a.e. on $[0, 1]$, and it's bounded by 1, so it's Riemann Integrable.

$$0 = R\int_{[0,1]} f_n \not\to R\int_{[0,1]} f$$

3 L^p Spaces

3.1 L^P Spaces

Recall

1. For $1 \leqslant p < \infty$, $(C([a,b]), \|\cdot\|_p)$ is a normed-vector space, where $\|f\|_p^p = \int_a^b |f|^p$
2. For $p = \infty$, $(C([a,b]), \|\cdot\|_\infty)$, $\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$ is a Banach Space.

Problem: $A \subseteq \mathbb{R}$ measurable, $1 \leqslant p < \infty$, $\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}}$ is NOT a norm on the vector space of integrable functions $f : A \to \mathbb{R}$. WHY? $\int_A |f|^p = 0 \iff f = 0$ a.e.

Definition 21

$A \subseteq \mathbb{R}$ measurable,

1. $M(A) = \{f : A \to \mathbb{R} \text{ measurable}\} \to$ vector space,

$$f \sim g \iff f = g \text{ a.e.}$$

   let $[f]$ represent the equivalence class.

2. $M(A)/\sim = \{[f] : f \in M(A)\}$. $\alpha[f] + \beta[g] = [\alpha f + \beta g]$ shows that it's a vector space.

Remark. If $f \sim g$ and f is integrable, then g is integrable and $\int_A f = \int_A g$

Definition 22

$A \subseteq \mathbb{R}$ measurable, $1 \leqslant p < \infty$,

$$L^p(A) = \left\{[f] \in M(A)/\sim : \int_A |f|^p < \infty\right\}$$

Remark. Suppose $[f], [g] \in L^p(A)$. Then $\int_A |f|^p, \int_A |g|^p < \infty$

1. $|f+g|^p \leqslant (|f| + |g|)^p \leqslant (2\max\{|f|, |g|\})^p \leqslant 2^p(|f|^p + |g|^p) \implies |f+g|^p$ integrable by comparison.
2. so $L^p(A)$ is a subspace of $M(A)/\sim$

Definition 23

$A \subseteq \mathbb{R}$ measurable,

$$L^\infty(A) = \{[f] \in M(A)/\sim : f \text{ bounded a.e.}\}$$

Remark.

1. $[f], [g] \in L^\infty(A)$

$$|f| \leqslant M \text{ off } B \subseteq A,\ m(B) = 0$$
$$|g| \leqslant N \text{ off } C \subseteq A,\ m(C) = 0$$

off $B \subseteq A$ means on $A \setminus B$.
For $x \notin B \cup C$,

$$|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant M + N$$

2. $L^\infty(A)$ is a subspace of $M(A)/\sim$

Proposition 64

$A \subseteq \mathbb{R}$ measurable, then

$$\|[f]\|_\infty = \inf\{M \geqslant 0 : |f| \leqslant M \text{ a.e.}\}$$

is a norm on $L^\infty(A)$

Remark.

1. $|f| \leqslant \|[f]\|_\infty + \frac{1}{n}$ off $m(A_N) = 0$, and $B = \cup_{n=1}^\infty A_n$ has measure 0
2. $|f| \leqslant \|f\|_\infty$ off B.

Proof.

1. $\|[f]\|_\infty = 0 \implies |f| \leqslant \|[f]\|_\infty$ a.e. $\implies |f| = 0$ a.e. $\implies f = 0$ a.e., then

$$[f] = [0]$$

in $L^\infty(A)$.

2. $|f| \leqslant \|[f]\|_\infty$ off B, $|g| \leqslant \|[g]\|_\infty$ off C. Off $B \cup C \implies$ measure 0:

$$|f + g| \leqslant |f| + |g| \leqslant \|[f]\|_\infty + \|[g]\|_\infty$$

By the definition of $\inf$, we have

$$\|[f + g]\|_\infty = \|[f] + [g]\|_\infty \leqslant \|[f]\|_\infty + \|[g]\|_\infty$$

$\square$

3.2 L^p Norm

Example 10

$p = 1$, $A \subseteq \mathbb{R}$ measurable, $[f], [g] \in L^1(A)$,

$$\begin{aligned} &|f+g| \leqslant |f| + |g| \\ \implies &\int_A |f+g| \leqslant \int_A |f| + \int_A |g| \\ \implies &\|f+g\|_1 \leqslant \|[f]\|_1 + \|[g]\|_1 \end{aligned}$$

Abusive Notation:

$$f \equiv [f] \in L^p(A)$$

Remember !
$f = g$ in $L^p(A)$ means $f = g$ a.e.

Definition 24

For $p \in (1, \infty)$ we define $q = \frac{p}{p-1}$ to be the Holder Conjugate of p.

Note:

1. $q = \frac{p}{p-1} \iff p = \frac{q}{q-1}$
2. $\frac{1}{p} + \frac{1}{q} = 1$

Definition 25

We define 1 and ∞ to be a pair of Holder conjugate.

Proposition 65: [Young's Inequality

$p, q \in (1, \infty)$ Holder conjugate. $\forall a, b > 0$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

Proof.

$$\begin{aligned}
f(x) &= \frac{1}{p}x^p + \frac{1}{q} - x \text{ on } (0,\infty)\\
f'(x) &= x^{p-1} - 1\\
f(1) &= \frac{1}{p} + \frac{1}{q} - 1 = 0\\
\implies f &\geqslant 0 \text{ on } (0,\infty)\\
\implies x &\leqslant \frac{1}{p}x^p + \frac{1}{q},\ \forall x > 0
\end{aligned}$$

Taking:

$$\begin{aligned}
x &= \frac{q}{b^{q-1}}\\
\implies \frac{a}{b^{q-1}} &\leqslant \frac{1}{p}\frac{a^p}{b^{(q-1)p}}\\
\implies \frac{a}{b^{q-1}} &\leqslant \frac{1}{p}\frac{a^p}{b^p} + \frac{1}{q}\\
\implies ab &\leqslant \frac{1}{p}a^p + \frac{1}{q}b^q
\end{aligned}$$

□

Proposition 66: [Holder's Inequality]

$A \subseteq \mathbb{R}$ measurable, $1 \leqslant p < \infty$, q is the Holder Conjugate. If $f \in L^p(A)$ and $g \in L^q(A)$ then $fg \in L^1(A)$ and $\int_A |fg| \leqslant \|f\|_p \|g\|_q$

Proof.

1. $p = 1, q = \infty$

   $$|fg| = |f||g| \leqslant |f|\|g\|_\infty \text{ a.e.}$$

   then $fg \in L^1(A)$ and

   $$\int_A |fg| \leqslant \int_A |f|\|g\|_\infty = \|g\|_\infty \|f\|_1$$

2. $1 < p < \infty$, q HC,

   $$|fg| = |f||g| \leqslant \frac{|f|^p}{p} + \frac{|g|^q}{q} \implies fg \in L^1(A)$$

   Also,

   $$\int_A |fg| \leqslant \frac{1}{p}\int_A |f|^p + \frac{1}{q}\int_A |g|^q = \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q$$

(a) $\|f\|_p = \|g\|_q = 1$,

$$\int_A |fg| \leqslant \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q$$

(b) $\frac{f}{\|f\|_p}, \frac{g}{\|g\|_q}$. By case a),

$$\frac{1}{\|f\|_p \|g\|_q} \int_A |fg| \leqslant 1 \implies \int_A |fg| \leqslant \|f\|_p \|g\|_q$$

□

Lemma 67

p, q HC, $f \in L^p(A)$. If $f \neq 0$,

$$f^* = \|f\|_p^{1-p} \mathrm{sgn(f)|f|^{p-1}}$$

is in $L^q(A)$ and

$$\int_A ff^* = \|f\|_p, \text{ and } \|f^*\|_q = 1$$

Proof.

1. $p = 1, q = \infty$

$$f^* = \mathrm{sgn(f)} \in \mathrm{L^\infty(A)}$$
$$\int_A ff^* = \int_A |f| = \|f\|_1, \ \|f^*\|_\infty = 1$$

2. $1 < p < \infty$, q HC

$$\int_A ff^* = \|f\|_p^{1-p} \int_A |f|^p = \|f\|_p^{1-p} \|f\|_p^p = \|f\|_p$$
$$\|f^*\|_q^q = \|f\|_p^{(1-p)q} \int_A |f|^{(p-1)q}$$
$$= \|f\|_p^{-p} \int_A |f|^p$$
$$= \|f\|_p^{-p} \|f\|_p^p = 1$$

□

Theorem 68: [Minkowski's Inequality]

$A \subseteq \mathbb{R}$ measurable and $1 \leqslant p < \infty$. If $f, g \in L^p(A)$ then

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$$

Proof. 1. $p = 1$ Done

2. $1 < p < \infty$

$$\begin{aligned}
\|f+g\|_p &= \int_A (f+g)(f+g)^* \\
&= \int_a f(f+g)^* + \int_A g(f+g)^* \\
&\underbrace{\leqslant}_{Holder} \|f\|_p \|(f+g)^*\|_q + \|g\|_p \|(f+g)^*\|_q \\
&= \|f\|_p + \|g\|_p
\end{aligned}$$

$\square$

3.3 Completeness

Theorem 69: [Riesz-Fisher]

For all measurable $A \subseteq \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, $L^P(A)$ is a Banach space.

Proof.

1. $p = \infty$, piazza

2. $1 \leqslant p < \infty$, Let $(f_n) \subseteq L^P(A)$ be strongly Cauchy Sequence. Therefore, there exists $(\varepsilon_n) \subseteq \mathbb{R}$ suCh that

   (a) $\|f_{n+1} - f_n\|_p \leqslant \varepsilon_n^2$

   (b) $\sum \varepsilon_n < \infty$

   Idea: Since $\mathbb{R}$ is complete, if $(f_n(x))$ is strongly-Cauchy then it converges. For each $n \in \mathbb{N}$,

$$\begin{aligned} A_n &= \{x \in A : |f_{n+1}(x) - f_n(x)| \geqslant \varepsilon_n\} \\ &= \{x \in A : |f_{n+1}(x) - f_n(x)|^p \geqslant \varepsilon_n^p\} \end{aligned}$$

   By Chebyshev's Inequality:

$$m(A_n) \leqslant \frac{1}{\varepsilon_n^p} \int_A |f_{n+1} - f_n|^p \leqslant \frac{1}{\varepsilon_n^p} \varepsilon_n^{2P} = \varepsilon_n^p$$
$$\sum m(A_n) \leqslant \sum \varepsilon_n^p \leqslant \left(\sum \varepsilon_n\right)^p < \infty$$

   which implies that $m(\limsup(A_n)) = 0$
   Fix $x \notin \limsup(A_n)$. Let $N = \max\{n : x \in A_n\}$. For $n > N$,

$$\begin{aligned} &|f_{n+1}(x) - f_n(x)| < \varepsilon_n^2, \ \sum \varepsilon_n < \infty \\ \implies &(f_n(x)) \text{ Cauchy} \\ \implies &f_n(x) \to f(x) \in \mathbb{R} \end{aligned}$$

   so $f_n \to f$ pointwise a.e.
   For $k \in \mathbb{N}$,

$$\|f_{n+k} - f_n\|_p \leqslant \|f_{n+k} - f_{n+k-1}\|_p + \ldots + \|f_{n+1} - f_n\|_p \leqslant \varepsilon_{n+k-1}^2 + \ldots + \varepsilon_n^2 \leqslant \sum_{i=n}^{\infty} \varepsilon_i^2$$

so $|f_{n+k} - f_n|^p \to |f_n - f|^p$ pointwise a.e. as $k \to \infty$.

By Fatou's Lemma,

$$\begin{aligned}
&\int_A |f_n - f|^p \\
\leqslant &\liminf_{k\to\infty} \int_A |f_{n+k} - f_n|^p \\
= &\liminf_{k\to\infty} \|f_{n+k} - f_n\|_p^p \\
\leqslant &\left[\sum_{i=n}^{\infty} \varepsilon_i^2\right]^p \to 0
\end{aligned}$$

so f_n converges w.r.t p-norm. $\square$

3.3.1 Separability:

Recall: A metric space X is separable if it has a countable, dense subset.

Example 11

$p = \infty$?
Suppose $\{f_n : n \in \mathbb{N}\}$ is dense in $L^\infty[0,1]$. For every $x \in [0,1]$, we may find

$$\|\chi_{[0,x]} - f_{\theta(x)}\|_\infty < \frac{1}{2}$$

For $x \neq y$ in $[0,1]$,

$$\|x_{[0,x]} - \chi_{[0,y]}\|_\infty = 1$$

so $\theta(x) \neq \theta(y)$ and $\theta[0,1] \to \mathbb{N}$ is injective, contradiction ($[0,1]$ not countable).

<u>Notation:</u>

- $Simp(A)$ = Simple functions on measure$A$
- $Step[a,b]$ = Step functions on$[a,b]$
- $Step_{\mathbb{Q}}[a,b]$ =Step functions on $[a,b]$ with rational partition (not including a, b) and functions values.

Proposition 70

$A \subseteq \mathbb{R}$ measurable, $1 \leqslant p < \infty$, $Simp(A)$ is dense in $L^P(A)$

Proof.

$$fr \in L^P(A) \to f \text{ measurable}$$

then there exists φ_n simple

1. $\varphi_n \to f$ pointwise
2. $|\varphi_n| \leqslant |f| \implies |\varphi_n|^p \leqslant |f|^p$

By comparison, $(\varphi_n) \subseteq L^P(A)$.
Note,

$$\|\varphi_n - f\|_p^p = \int_A |\varphi_n - f|^p$$
$$|\varphi_n - f|^p \leqslant 2^p(|\varphi_n|^p + |f|^p)$$
$$\leqslant 2^{p+1}|f|^p$$

so by the Lebesgue Dominate Convergence Theorem

$$\lim_{n\to\infty} \|\varphi_n - f\|_p^p = \lim_{n\to\infty} \int_A |\varphi_n - f|^p = \int 0 = 0$$

□

Fact: the above proposition is true for $p = \infty$ (but it's not seperable).

Proposition 71

$1 \leqslant p < \infty$. $Step[a,b]$ is dense in $L^P[a,b]$

Proof. $A \subseteq [a,b]$ measurable, $\chi_A[a,b] \to \mathbb{R}$.
Littlewood 1:$\exists \dot{\cup}_{i=1}^{n} I_i = U$, where I_is are bounded open intervals. And $m(U \triangle A < \varepsilon$ and $\chi_U : [a,b] \to \mathbb{R}$ is a step function.

$$\begin{aligned} &\|\chi_U - \chi_A\|_p^p \\ =&\int_A \|\chi_U - \chi_A|^p \\ =&\int_{U\triangle A} 1^p \\ =&m(U\triangle A) \\ \implies &\|\chi_U - \chi_A\|_p < \varepsilon \end{aligned}$$

so for all characteristic function, we can approach as close as we want by a step function. Simple function is just made of **finitely** many characteristic functions. ☐

Corollary 72

$1 \leqslant p < \infty$. $Step_{\mathbb{Q}}[a,b]$ is dense in $L^p[a,b]$ (step functions are dense, so for each step function, you can modify the function a little bit by rationals). Therefore, $L^p[a,b]$ is separable.

Proposition 73

$1 \leqslant p < \infty$, $L^p(\mathbb{R})$ is separable.

Proof. $1 \leqslant p < \infty$, $L^p(\mathbb{R})$ is separable.

$$F_n = \left\{ f \in L^p(\mathbb{R}) | f|_{[-n,n]} \in Step_{\mathbb{Q}}[-n,n], f|_{\mathbb{R}\setminus[-n,n]} = 0 \right\}$$

$F = \cup_{n=1}^{\infty} F_n$ countable. Take $f \in L^p(\mathbb{R})$. Fix $n \in \mathbb{N}$, we have $f|_{[-n,n]} \in L^p([-n,n])$ We show

$$f\chi_{[-n,n]} \to f \text{ in } L^p(\mathbb{R})$$

Note:

1.

$$\begin{aligned} &\|f\chi_{[-n,n]} - f\|_p^p \\ =&\int_{\mathbb{R}} |f\chi_{[-n,n]} - f|^p \\ =&\int_{\mathbb{R}\setminus[-n,n]} |f|^p \\ =&\int_{\mathbb{R}} |f|^p \chi_{\mathbb{R}\setminus[-n,n]} \end{aligned}$$

2. $||f|^p \chi_{\mathbb{R}\setminus[-n,n]}| \leqslant |f|^p$ which is integrable

3. By the Lebesgue Dominated Convergence Theorem

$$\begin{aligned} &\lim_{n\to\infty} \|f\chi_{[-n,n]} - f\|_p^p \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} |f\chi_{[-n,n]} - f|^p = \int_{\mathbb{R}} 0 = 0 \end{aligned}$$

so $\|f\chi_{[-n,n]} - f\|_p \to 0$
For each $n \in \mathbb{N}$, $\exists \varphi_n \in F$ such that $\|f\chi_{[-n,n]} - \varphi_n\|_p < \frac{1}{n}$, so

$$\|\varphi_n - f\|_p \to 0$$

□

Theorem 74

$1 \leqslant p < \infty$, $A \subseteq \mathbb{R}$ measurable, $L^p(A)$ is separable.

Proof. F as before, $\{f|_A \colon f \in F\}$ is a countable dense subset of $L^p(A)$ □

4 Fourier Analysis

4.1 Hilbert Space

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Definition 26

V is a vector space over $\mathbb{F}$. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that

1. $\forall v \in V$, $\langle v, \rangle \in \mathbb{F}$, $\langle v, v \rangle \geqslant 0$ with $\langle v, v \rangle = 0$ if and only $v = 0$

2. $\forall v, w \in V$,
$$\langle v, w, \rangle = \overline{\langle w, v \rangle}$$

3. $\forall \alpha \in \mathbb{F}$, $u, v, w \in V$,
$$\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$$

   We call $(V, \langle \cdot, \cdot \rangle$ an inner product space.

Proposition 75

Let V be an inner product space. Then

$$\|v\| = \sqrt{\langle v, v \rangle}$$

is a norm on V. We call $\|\cdot\|$ the norm induced by $\langle \cdot, \cdot \rangle$

Example 12

$A \subseteq \mathbb{R}$ measurable. $V = L^2(A)$, $\langle f, g \rangle = \int_A fg$ is an inner product space.
Note: $\sqrt{\langle f, f \rangle} = \left(\int_A |f|^2\right)^{\frac{1}{2}} = \|f\|_2$

Example 13

$A \subseteq \mathbb{R}$ measurable. $V = L^2(A, \mathbb{C})$, $\langle f, g \rangle = \int_A f\overline{g}$ and $\sqrt{\langle f, f \rangle} = \|f\|_2$

Proposition 76: [Parallelogram Law]

Let V be an inner product space. $\forall u, v \in V$,

$$\|u + v\|^2 + \|u - v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

Proof.

$$\begin{aligned}
&\|u+v\|^2+\|u-v\|^2 \\
=&\langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
=&\langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle + \langle u,u\rangle - 2\langle u,v\rangle + \langle v,v\rangle \\
=&2\|u\|^2+2\|v\|^2 \\
=&2\left(\|u\|^2+\|v\|^2\right)
\end{aligned}$$

$\square$

Example 14

$1 \leqslant p < \infty$, $V = L^p[0,2]$ and $f = \chi_{[0,1]}, g = \chi_{[1,2]}$

$$\begin{aligned}
\|f\|_p^2 &= \left(\int_{[0,2]} |f|^p\right)^{\frac{2}{p}} \\
&= 1^{\frac{2}{p}} = 1 \\
\|g\|_p^2 &= 1^{\frac{2}{p}} = 1 \\
\|f+g\|_p^2 &= 2^{\frac{2}{p}} \\
\|f-g\|_p^2 &= 2^{\frac{2}{p}}
\end{aligned}$$

so by Parallelogram Law

$$2^{\frac{2}{p}} + 2^{\frac{2}{p}} = 2(1+1) \iff 2^{\frac{2}{2}} = 2 \iff p = 2$$

so $\|\cdot\|_p$ is induced by an inner product if and only if $p = 2$. You can also show that $\|\cdot\|_\infty$ is not induced by an inner product.

Definition 27

A Hilbert Space is a complete inner product space (i.e. A Banach Space whose norm is induced by an inner product).

Example 15

$L^2(A), L^2(A, \mathbb{C})$ are Hilbert Spaces.

4.2 Orthogonality

Definition 28

Let V be an inner product space. We say $v, w \in V$ are orthogonal if $\langle v, w\rangle = 0$.

Example 16

$f, g \in L^2\left([-\pi, \pi], \mathbb{C}\right),\ m \neq n,\ f(x) = e^{inx},\ g(x) = e^{imx}$, then

$$
\begin{aligned}
\langle f, g\rangle &= \int_{[-\pi,\pi]} f\overline{g} \\
&= \int_{[-\pi,\pi]} e^{inx}e^{-imx}dx \\
&= \int_{[-\pi,\pi]} e^{ix(n-m)}dx \\
&= \int_{[-\pi,\pi]} \cos((n-m)x) + i\int_{[-\pi,\pi]} \sin((n-m)x) \\
&= R\int_{-\pi}^{\pi} \cos((n-m)x) + iR\int_{-\pi}^{\pi} \sin((n-m)x)dx \\
&= 0
\end{aligned}
$$

Theorem 77: [Pythagorean Theorem]

Let V be an inner product space. If $v_1, \ldots, v_n \in V$ are pairwise orthogonal, then,

$$\left\|\sum V_i\right\|^2 = \sum \|V_i\|^2$$

Definition 29

Let V be an inner product space. We say $A \subseteq V$ is <u>orthonormal</u> if the elements of A are pairwise orthogonal and $\|v\| = 1, \forall v \in A$.

Corollary 78

Let V be an inner product space, $\{v_1, \ldots, v_n\}$ orthonormal,

$$\left\|\sum \alpha_i v_i\right\|^2 = \sum |\alpha_i|^2$$

Example 17

$L^2([-\pi,\pi],\mathbb{C})$, $A = \left\{\frac{1}{\sqrt{2\pi}}e^{inx} : n \in \mathbb{Z}\right\} \implies$ pairwise orthogonal.

$$\begin{aligned}\frac{1}{2\pi}\left\|e^{inx}\right\|_2^2 \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} e^{inx}e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} 1 = 1\end{aligned}$$

so A is orthonormal

Definition 30

Let V be an inner product space. An orthonormal basis is a maximal (w.r.t $\subseteq$) orthonormal subset of V. (Note it might not ba basis).

Fact: An inner product space always has an orthonormal basis.

Fact: Let H be a Hilbert space. If $W \subseteq H$ is closed subspace then there exists a subspace $W^\perp \subseteq H$ such that

$$H = W \oplus W^\perp$$

and $\langle w, z\rangle = 0$ for all $w \in W$ and $z \in W^\perp$.

Theorem 79

Let H be a Hilbert space, then H has a countable ONB (orthonormal basis) if and only if H is separable.

Proof.

- $\implies$ Let be B be a countable orthonormal basis for H.
  Claim: $w = \mathrm{Span(B)}$, $\overline{w} = H$
  Suppose $\overline{w} \neq H$. Since $H = \overline{w} \oplus \overline{w}^\perp$. We may find $0 \neq x \in \overline{w}^\perp$. We may assume $\|x\| = 1$. so $B \cup \{x\}$ is orthonormal. Contradiction! So $\overline{w} = H$.
  We can also show that $\overline{\mathrm{Span}_{\mathbb{Q}}(\mathrm{B})} = H$ where $\mathrm{Span}_{\mathbb{Q}}(\mathrm{B})$ is the span of B only using rational numbers as the coefficients. Hence, H is separable.

- $\impliedby$ Suppose H doesn't have an orthonormal basis which is countable. Let B be ONB for H, so B is uncountable.
  For $u \neq v$ in B,
  $$\|u - v\|^2 = \|u\|^2 + \|v\|^2 = 2 \implies \|u - v\| = \sqrt{2}$$

Suppose $X \subseteq H$ such that $\overline{X} = H$. $\forall u \in B$, there exists $x_n \in X$ such that

$$\|x_n - u\| < \frac{\sqrt{2}}{2}$$

but for $u \neq v$ in B, we have that

$$x_u \neq x_v$$

so

$$\varphi : B \mapsto X,\ \varphi(u) = x_u$$

is an injection. So X is uncountable because B is uncountable, so H is not separable, contradiction.

$\square$

Example 18

$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z} \right\}$ is a countable orthonormal set in $L^2([-\pi, \pi], \mathbb{C})$. We can clearly see that it is countable, orthonormal, but what about maximal?

4.3 Big Theorems

Remark. Let H be an inner product space, $\{v_1, \ldots, v_n\}$ orthonormal.
If $v = \sum \lambda_i v_i$ then $\lambda_i = \langle v, v_i \rangle$. We call $\langle v, v_i \rangle$ the Fourier Coefficients of v w.r.t. $\{v_1, \ldots, v_n\}$

Definition 31

Let H be a Hilbert space, $\{v_1, v_2, \ldots\}$ orthonormal. For $v \in H$, we call

$$\sum_{i=1}^{\infty} \langle v, v_i \rangle v_i$$

the Fourier Series of v relative to $\{v_1, v_2, \ldots\}$ and write

$$v \sim \sum_{i=1}^{\infty} \langle v, v_i \rangle v_i$$

- Does this series converge?
- Does it converge to v?

Theorem 80: [Best Approximation]

Let H be a Hilbert Space, $\{v_1, \ldots, v_n\}$ orthonormal. For $v \in H$, $\|v - \sum \lambda_i v_i\|$ is minimized when $\lambda_i = \langle v, v_i \rangle$
Moreover,

$$\left\| v - \sum \langle v, v_i \rangle v_i \right\|^2 = \|v\|^2 - \sum |\langle v, v_i \rangle|^2$$

Proof.

1. $W = \mathrm{Span}\{\mathrm{v}_1, \ldots, \mathrm{v}_\mathrm{n}\}$ closed, $v = W \oplus W^\perp$
2. $x \in W,\ v = w + z,\ w \in W, z \in W^\perp$,

   $$\|v - x\|^2 = \|w + z - x\|^2 = \|w - x + z\|^2 = \|w - x\|^2 + \|z\|^2 \geqslant \|z\|^2 = \|v - x\|^2$$

   so $\|v - x\| \geqslant \|v - w\|$, the closet point in W to v is w, the orthonormal projection.
3. $v = \sum \lambda_i v_i + z,\ z \in W^\perp$,

   $$\langle v, v_i \rangle = \lambda_i + \underbrace{\langle z, v_i \rangle}_{0} = \lambda_i$$

4. $v = \sum \langle v, v_i \rangle v_i + z,\ z \in W^\perp$, then

   $$\begin{aligned} \|v\|^2 &= \left\| \sum \langle v, v_i \rangle v_i \right\|^2 + \|z\|^2 \\ &= \sum |\langle v, v_i \rangle|^2 + \|z\|^2 \end{aligned}$$

so,

$$\left\|v - \sum \langle v, v_i \rangle v_i\right\|^2 = \|z\|^2 = \|v\|^2 - \sum |\langle v, v_i \rangle|^2$$

$\square$

Theorem 81: [Bessel's Inequality]

Let H be a Hilber Space, $\{v_1, \dots, v_n\}$ be orthonormal. If $v \in H$,

$$\sum_{i=1}^{n} |\langle v, v_i \rangle|^2 \leqslant \|v\|^2$$

Proof.

$$\|v\|^2 - \sum |\langle v, v_i \rangle|^2 = \left\|v - \sum \langle v, v_i \rangle v_i\right\|^2 \geqslant 0$$

$\square$

Theorem 82: [Parseral's Identity]

Let H be a Hilbert space, $\{v_1, v_2, \dots\}$ orthonormal.
For $v \in H$,

$$\sum_{i=1}^{\infty} |\langle v, v_i \rangle|^2 = \|v\|^2 \iff \lim_{n\to\infty} \left\|v - \sum_{i=1}^{n} \langle v, v_i \rangle v_i\right\|^2 = 0$$

Theorem 83: [Orthonormal Basis Test]

Let H be a separable Hilbert Space $\{v_1, v_2, \dots\}$ orthonormal. TFAE:

1. $\{v_1, v_2, \dots\}$ is an orthonormal basis.
2. $\overline{\mathrm{Span}\{v_1, v_2, \dots\}} = H$
3. $\lim_{n\to\infty} \|v - \sum_{i=1}^{n} \langle v, v_i \rangle v_i\| = 0, \ \forall v \in H$

Proof.

- (1) $\implies$ (2) Done
- (2) $\implies$ (3)
  If $\{v_1, v_2, \dots\}$ is not maximal then we may find $u \in H$, $\|u\| = 1$ such that $\langle u, v_i \rangle = 0, \ \forall i \in \mathbb{N}$. Since $C = \{x \in H : \langle x, u \rangle = 0\}$ is closed, $u \notin \overline{\mathrm{Span}\{v_1, v_2, \dots\}}$ ($u \notin C, \langle u, u \rangle = 1$, $\overline{\mathrm{Span}\{v_1, v_2, \dots\}} \subseteq C$).

- $(2) \implies (3)$
  Let $v \in H$ and let $\varepsilon > 0$ be given. Let $\sum_{i=1}^{N} \alpha_i v_i \in \mathrm{Span}\{\mathrm{v}_1, \ldots\}$ such that

$$\left\| v - \sum_{i=1}^{N} \alpha_i v_i \right\| < \varepsilon$$

  so $\left\| v - \sum_{i=1}^{N} \langle v, v_i \rangle v_i \right\| < \varepsilon$.
  For $n \geqslant \mathbb{N}$,

$$\begin{aligned} &\left\| v - \sum_{i=1}^{n} \langle v, v_i \rangle v_i \right\| \\ \leqslant &\left\| v - \sum_{i=1}^{N} \langle v, v_i \rangle v_i \right\| + \left\| \sum_{i=N+1}^{n} \langle v, v_i \rangle v_i \right\| \\ < &\varepsilon + \sqrt{\sum_{N+1}^{\infty} |\langle v, v_i \rangle|^2} \longrightarrow 0 \text{ as } N \to \infty \end{aligned}$$

  because by Bessel's Inequality, $\sum_{i=1}^{N} |\langle v, v_o \rangle|^2$ is a bounded increasing sequence, so $\sum_{N+1}^{\infty} |\langle v, v_i \rangle|^2$ will go to 0.

- $(3) \implies (2)$, similar.

$\square$

4.4 Fourier Series

1. Is $\left\{\frac{1}{\sqrt{2\pi}}e^{inx} : n \in \mathbb{Z}\right\}$ an ONB for $L^2([-\pi,\pi],\mathbb{C})$?

2. Is $\mathrm{Span}\left\{\mathrm{e}^{\mathrm{inx}} : \mathrm{n} \in \mathbb{Z}\right\}$ dense in $Ł^2([-\pi,\pi],\mathbb{C})$?

3. Is $\mathrm{Span}\left\{\mathrm{e}^{\mathrm{inx}} : \mathrm{n} \in \mathbb{Z}\right\}$ dense in $L^1([-\pi,\pi],\mathbb{C})$

Definition 32

Let $T = [-\pi,\pi)$. We call T the Torus or the circle. We define.

$$L^p(T) = L^p([-\pi,\pi],\mathbb{C})$$

for $1 \leqslant p < \infty$.
Using the norm,

$$\|f\|_p = \left(\frac{1}{2\pi}\int_T |f|^p\right)^{\frac{1}{p}}$$

$L^p(T)$ is a separale Banach Space.

Remark.

1. As a group under addition module 2π,

$$T \cong \mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C} : |z| = 1\}$$

2. In this way, T is a locally compact abelion group.

3. There is a one-to-one correspondence between

$$f : T \mapsto \mathbb{C}$$

and 2π-periodic function

$$f : \mathbb{R} \mapsto \mathbb{C}$$

Definition 33

$f \in L^1(T)$

1. We define the n^{th} ($n \in \mathbb{Z}$) Fourier Coefficients of f by

$$\langle f, e^{inx} \rangle := \frac{1}{2\pi} \int_T f(x) e^{-inx} dx$$

2. We define the Fourier Series of f by

$$f \sim \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

where $a_n = \langle f, e^{inx} \rangle$.

3. We let

$$S_N(f, x) = \sum_{-N}^{N} a_n e^{inx}$$

denote the N^{th} partial sum of the above Fourier series.

Proposition 84

Consider the trignometric polynomial $f \in L^1(T)$ given by

$$f(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

for some $a_i \in \mathbb{C}$.
For each $-N \leqslant n \leqslant N$,

$$\langle f, e^{inx} \rangle = a_n$$

Why?

$$\frac{1}{2\pi} \int_T e^{imx} e^{-inx} dx = \delta_{m,n} = \begin{cases} 1, m = n \\ 0, m \neq n \end{cases}$$

Remark. Suppose $f \in L^1(T)$ is real-valued, $f \sim \sum_{n \in \mathbb{Z}} a_n e^{inx}$.

For $N \in \mathbb{N}$,

$$\begin{aligned} S_N(f,x) &= \sum_{n=-N}^{N} a_n e^{inx} \\ &= a_0 + \sum_{n=1}^{N} (a_n e^{inx} + a_{-n} e^{-inx}) \\ &= a_0 + \sum_{n=1}^{N} (a_n + a_{-n})\cos(nx) + i(a_n - a_{-n})\sin(nx) \\ \\ &= a_0 + \sum_{n=1}^{N} b_n \cos(nx) + c_n \sin(nx) \end{aligned}$$

Now,

$$a_0 = \frac{1}{2\pi}\int_T f(x)e^{-i0x}dx = \frac{1}{2\pi}\int_T f(x)dx$$

$$\begin{aligned} b_n &= a_n + a_{-n} \\ &= \frac{1}{2\pi}\int_T f(x)(e^{-inx} + e^{inx})dx \\ &= \frac{1}{\pi}\int_T f(x)\cos(nx)dx \end{aligned}$$

$$\begin{aligned} c_n &= i(a_n - a_{-n}) \\ &= \frac{i}{2\pi}\int_T f(x)(e^{-inx} - e^{inx})dx \\ &= \frac{1}{\pi}\int_T f(x)\sin(nx)dx \end{aligned}$$

are all real-valued.

4.5 Fourier Coefficients

Proposition 85

$f, g \in L^1(T)$

1. $\langle f+g, e^{inx}\rangle = \langle f, e^{inx}\rangle + \langle g, e^{inx}\rangle$
2. For $\alpha \in \mathbb{C}$, $\langle \alpha f, e^{inx}\rangle = \alpha \langle f, e^{inx}\rangle$
3. If $\overline{f} : T \mapsto \mathbb{C}$ is defined by $\overline{f}(x) = \overline{f(x)}$, then $\overline{f} \in L^1(T)$ and $\langle \overline{f}, e^{inx}\rangle = \overline{\langle f, e^{inx}\rangle}$

Proof.

1. Trivial
2. Trivial
3. $|f| = |\overline{f}| \implies \overline{f} \in L^1(T)$,

$$
\begin{aligned}
&\left\langle \overline{f}, e^{inx} \right\rangle \\
=&\frac{1}{2\pi}\int_T \overline{f}(x)e^{-inx}dx \\
=&\frac{1}{2\pi}\int_T \overline{f(x)e^{inx}}dx \\
=&\frac{1}{2\pi}\int_T Re(\overline{f(x)e^{inx}})dx + \frac{i}{2\pi}\int_T Im(\overline{f(x)e^{inx}})dx \\
=&\frac{1}{2\pi}\int_T Re(f(x)e^{inx})dx - \frac{i}{2\pi}\int_T Im(f(x)e^{inx})dx \\
=&\overline{\frac{1}{2\pi}\int_T f(x)e^{inx}dx} \\
=&\overline{\langle f, e^{-inx} \rangle}
\end{aligned}
$$

□

Proposition 86

$f \in L^1(T)$, $\alpha \in \mathbb{R}$. By a previous remark, we may view $f : \mathbb{R} \mapsto \mathbb{C}$ as a 2π-periodic function which is integrable over T. For $\alpha \in \mathbb{R}$, $f_\alpha : \mathbb{R} \mapsto \mathbb{C}$ given by $f_\alpha(x) = f(x-\alpha)$ is integrable over T and $\langle f_\alpha, e^{inx} \rangle = \langle f, e^{inx} \rangle e^{-in\alpha}$

Proposition 87

$f \in L^1(T)$. $\forall n \in \mathbb{Z}$, $|\langle f, e^{inx} \rangle| \leqslant \|f\|_1$

Proof.

$$
\begin{aligned}
|\left\langle f, e^{inx} \right\rangle| &= \left|\frac{1}{2\pi}\int_T f(x)e^{-inx}dx\right| \\
&\leqslant \frac{1}{2\pi}\int_T \left|f(x)e^{-inx}\right| dx \\
&= \frac{1}{2\pi}\int_T |f(x)|dx
\end{aligned}
$$

□

Corollary 88

$f_k \mapsto f$ in $L^1(t)$,
$$\forall n \in \mathbb{Z},\ \left\langle f_k, e^{inx} \right\rangle \mapsto \left\langle f, e^{inx} \right\rangle$$

Proof.

$$\begin{aligned} &\left|\left\langle f_k, e^{inx} \right\rangle - \left\langle f, e^{inx} \right\rangle\right| \\ =&\left|\left\langle f_k - f, e^{inx} \right\rangle\right| \\ \leqslant&\|f_k - f\|_1 \longrightarrow 0 \end{aligned}$$

□

Remark. Let $\mathrm{Trig}(\mathrm{T})$ denote the set of Trigonometric polynomials on T. By A3, $\overline{\mathrm{Trig}(\mathrm{T})} = L^1(T)$

Theorem 89: [Riemann-Lebesgue Lemma]

If $f \in L^1(T)$, then
$$\lim_{|n|\to\infty} \left\langle f, e^{inx} \right\rangle = 0$$

Proof. Let $\varepsilon > 0$ be given and let $P \in \mathrm{Trig}(\mathrm{T})$ such that $\|f - P\|_1 \leqslant \varepsilon$. Say $P(x) = \sum_{k=-N}^{N} a_k e^{ikx}$. For $|n| > N$, we have that $\langle P, e^{inx} \rangle = 0$, so

$$\left|\left\langle f, e^{inx} \right\rangle\right| = \left|\left\langle f - P, e^{inx} \right\rangle\right| \leqslant \|f - P\|_1 < \varepsilon$$

□

4.6 Vector-Valued Integration

Definition 34

Let B be a Banch space and let $f : [a, b] \to B$ be a function. Consider a partition $P : a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$. We define a Riemann sum of f over P by

$$S(f, P) = \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}) \in B$$

where each $t_i^* \in [t_{i-1}, t_i]$.

Definition 35

Let B and f Be as above. We say f is Riemann Integrable if there exists $z \in B$ such that $\forall \varepsilon > 0$, there is a partition P_ε of $[a, b]$ such that whenever P is a refinement of P_ε and $S(f, p)$ is a Riemann sum then

$$\|S(f, P) - z\| < \varepsilon$$

We call z the integral of f over $[a, b]$ and write $z = R\int_a^b f(x)dx$.

A natural question to ask would be: Why are we doing this only for Banach Space?

Theorem 90: [Cauchy Criterion]

Let B be a Banach space and let $f : [a, b] \to B$ be a function. Then f is Riemann Integrable if and only if $\forall \varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that whenever P and Q are refinements of P_ε we have,

$$\|S(f, p) - S(f, Q)\| < \varepsilon$$

for any Riemann sums $S(f, P)$ and $S(f, Q)$.

Proof. Suppose f is Riemann integrable with $z = R\int_a^b f(x)dx$. Let $\varepsilon > 0$ be given. We may find a partition $P_{\varepsilon/2}$ such that whenever P is a refinement partition of $P_{\varepsilon/2}$ then

$$\|S(f, P) - S(f, Q)\| \leqslant \|S(f, P) - z\| + \|z - S(f, Q)\| < \varepsilon$$

Conversely, assume the Cauchy Criterion holds. In particular, for each $n \in \mathbb{N}$, we may find a partition P_n of $[a, b]$ which corresponds to $\varepsilon = \frac{1}{n}$, as per Cauchy Criterion. Without loss of generality, we may assume that each P_{n+1} is a refinement of P_n. For each $n \in \mathbb{N}$, let $S(f, P_n)$ be a Riemann sum. Let $\varepsilon > 0$ be given. Choosing $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$, we see that for $m, n \geqslant \mathbb{N}$,

$$\|S(f, P_m) - S(f, P_n)\| < \frac{1}{N} < \varepsilon$$

Since B is a Banach Space, $S(f, P_n) \to z \in B$

We claim that f is Riemann Integrable with $R\int_a^b dx = z$. Let N and P_N be as above. Moreover, we know $\exists M > N$ such that $\|S(f, P_M) - z\| < \frac{\varepsilon}{2}\|$. Now if P is any refinement partition of P_N, then

$$\|S(f, P) - z\| \leqslant \|S(f, P) - S(f, P_M)\| + \|S(f, P_M) - z\| < \varepsilon$$

$\square$

Theorem 91

If B is a Banach Space and $f : [a, b] \to B$ is continuous, then f is Riemann integrable.

4.7 Summability Kernels

Definition 36

$f, g \in L^1(T)$. The convolution of f and g is the functions

$$f * g : T \mapsto \mathbb{C}$$

given by

$$(f * g)(x) = \frac{1}{2\pi} \int_T f(t) g(x-t) dt = \frac{1}{2\pi} \int_T f(t) g_t(x) dt$$

Facts:

1. Given $f, g \in L^1(T)$, $f * g \in L^1(T)$ as well.
2. $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$
3. This means $L^1(T)$ a Banach Algebra (Banach Space with continuous multiplication, we can think convolution as a "multiplication").

Let $C(T)$ denote the set of continuous functions $T \to \mathbb{C}$

Definition 37

A summability kernel is a sequence $(K_n) \subseteq C(T)$ such that

1. $\frac{1}{2\pi} \int_T K_n = 1$
2. $\exists M$, $\forall n$, $\|K_n\|_1 \leqslant M$
3. $\forall 0 < \delta < \pi$,
$$\lim_{n \to \infty} \left(\int_{-\pi}^{-\delta} |K_n| + \int_{\delta}^{\pi} |K_n| \right) = 0$$
This means summability kernels are concentrated at 0.

Proposition 92

Let $(B, \|\cdot\|_B)$ be a Banach Space (with scaler $\mathbb{C}$. Let $\varphi : T \mapsto B$ be continuous. Let $(K_n) \subseteq C(t)$ be a summability kernel. Then

$$\lim_{n \to \infty} \frac{1}{2\pi} \underbrace{\int_T K_n(t) \varphi(t) dt}_{\text{Riemann vector-valued integral}} = \varphi(0)$$

in the B-norm.

Proof. Appendix using $(2), (3)$ □

Remark. $\varphi : T \to L^1(T)$, given by

$$\varphi(t) = f_t = f(x-t)$$

is continuous.

Theorem 93

$f \in L^1(T)$, K_n is a summability kernel. In $L^1(T)$,

$$f = \lim_{n\to\infty} K_n * f$$

Proof. Let $\varphi(t) = f(x-t)$

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2\pi} \int_T K_n(t)\varphi(t)dt = \varphi(0) \\
\implies &\lim_{n\to\infty} \frac{1}{2\pi} \int_T K_n(t) f(x-t)dt = \varphi(0) = f(x-0) = f(x) \\
\implies &\lim_{n\to\infty} (K_n * f)(x) = f(x)
\end{aligned}$$

□

4.8 Dirichlet Kernel

We want to find (K_n) such that $K_n * f = S_n(f)$, which is the n^{th} partial sum of Fourier Series of f.

Remark. Let $f \in L^1(T)$. For $n \in \mathbb{Z}$ consider

$$\varphi_n(x) = e^{inx} \in L^1(T)$$

Then

$$\begin{aligned}
&(\varphi_n * f)(x)\\
=&\frac{1}{2\pi}\int_T \varphi_n(t) f_t(x)dt\\
=&\frac{1}{2\pi}\int_T e^{int} f(x-t)dt\\
=&\frac{1}{2\pi}e^{inx}\int_T e^{-in(x-t)} f(x-t)dt\\
=&\frac{1}{2\pi}e^{inx}\int_T e^{-in(-t)} f(-t)dt\\
=&\frac{1}{2\pi}e^{inx}\int_T e^{-int} f(t)dt\\
=&e^{inx}\left\langle f, e^{inx}\right\rangle
\end{aligned}$$

Remark. $f \in L^1(T)$, if $P(x) = \sum_{k=-n}^{n} a_k e^{ikx}$, then

$$\begin{aligned}
&(P * f)(x)\\
=&\frac{1}{2\pi}\int_T P(t) f(x-t)dt\\
=&\sum_{k=-n}^{n}\frac{a_n}{2\pi}\int_T e^{ikt} f(x-t)dt\\
=&\sum_{k=-n}^{n} a_n(\varphi_n * f)(x)\\
=&\sum_{k=-n}^{n} a_n e^{ikx}\left\langle f, e^{ikx}\right\rangle
\end{aligned}$$

Definition 38

$D_n(x) = \sum_{k=-n}^{n} e^{ikx}$ is the Dirichlet Kernel of order n. And

$$\begin{aligned}&(D_n * f)(x)\\ =&\sum_{k=-n}^{n} e^{ikx}\left\langle f, e^{ikx}\right\rangle\\ =&S_n(f,x)\end{aligned}$$

which is the n^{th} partial sum we want.However, it's NOT a summability kernel.

4.9 Fejér Kernel

Idea: $(x_n) \subseteq \mathbb{C}$, consider

$$y_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

Exer: If $x_n \to x$, then $y_n \to y$.

Definition 39

The Fejér Kernel of order n is

$$F_n(x) = \frac{D_0(x) + D_1(x) + \ldots + D_n(x)}{n+1}$$

Remark.

$$F_0(x) = D_0(x) = 1$$
$$F_1(x) = \frac{e^{-x} + 2e^{i0x} + e^{ix}}{2}$$
$$F_2(x) = \frac{e^{-2x} + 2e^{-x} + 3e^{i0x} + 2e^{ix} + e^{i2x}}{3}$$
$$\vdots$$
$$F_n(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}$$

Remark. (F_n) is a summability kernel.

Definition 40

$$\begin{aligned} F_n * f &= \frac{1}{n+1} \sum_{k=0}^{n} D_k * f \\ &= \frac{1}{n+1} \sum_{k=0}^{n} S_k(f) \\ &= \frac{S_0(f) + S_1(f) + \ldots + S_n(f)}{n+1} \\ &=: \sigma_n(f) \end{aligned}$$

which is the n^{th} Cesaro mean.

Theorem 94

$f \in L^1(T)$, (F_n) Fejér.

$$\begin{aligned}&\lim_{n\to\infty} F_n * f\\ =&\lim_{n\to\infty} \sigma_n(f)\\ =&f\end{aligned}$$

in $L^1(T)$.

Remark. If $(S_n(f))$ converges in $L^1(T)$ then $S_n(f) \to f$ in $L^1(T)$.

4.10 Fejér's Theorem

<u>Idea:</u> L^1 convergence is great theoretically, but pointwise convergence is practical.

Theorem 95: [Fejér's Theorem]

For $f \in L^1(T)$ and $t \in T$ consider

$$\omega_f(t) = \frac{1}{2} \lim_{x \to 0^+} (f(t+x) + f(t-x))$$

provided the limit exists, then

$$\sigma_n(f,t) \to \omega_f(t)$$

In particular, if f is continuous at t then

$$\sigma_n(f,t) \to f(t)$$

In practice:

1. Fix $x \in T$
2. Prove $(S_n(f,x))$ converged
3. Then
$$S_n(f,x) \to \omega_f(x)$$
4. If f is continuous at x then $S_n(f,x) \to f(x)$, i.e. $S(f,x) = f(x)$.

Example 19

$f \in L^1(T),\ f(x) = |x|,$

$$S_n(f,x) = a_0 + \sum_{k=1}^{n}(b_k \cos(kx) + c_k \sin(kx))$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| dx = \frac{\pi}{2} \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(kx) dx \\ &= \frac{2(-1)^k - 2}{k^2\pi} \\ c_k &= \frac{1}{\pi}\int_{\pi}^{\pi} |x| \sin(kx) dx = 0 \end{aligned}$$

so

$$\begin{aligned} &S_n(f,x) \\ =&\frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{(-1)^k - 1}{k^2}\cos(kx)\right) \\ =&\frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{(n+1)/2}\left(\frac{-2}{(2k-1)^2}\cos((2k-1)x)\right) \end{aligned}$$

Note: $(S_n(f,x))$ converges by comparison with $\sum \frac{1}{(2x-1)^2}$.
Since f is continuous,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)^2}$$

1. Taking $x = 0$:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \implies \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

2.

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^2} &= \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \\
&= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{\pi^2}{8} \\
\implies \quad & \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}
\end{aligned}
$$

4.11 Homogeneous Banach Space

Definition 41

A homogeneous Banach Space is a Banach Space $(B, \|\cdot\|_b)$ such that

1. B is a subspace of $L^1(T)$
2. $\|\cdot\|_1 \leqslant \|\cdot\|_b$
3. $\forall f \in B, \forall \alpha \in T, \|f_\alpha\|_B = \|f\|_B$ (assuming $f_\alpha \in B$).
4. $\forall f \in B, \forall t_0 \in T,$
$$\lim_{t \to t_0} \|f_t - f_{t_0}\|_B = 0$$

Example 20

$(L^p(T), \|\cdot\|_p)$ ($p < \infty$).

Theorem 96

Let B be a homogeneous Banach Space (K_n) summability kernel. $\forall f \in B$,

$$\lim_{n \to \infty} \|K_n * f - f\|_B = 0$$

Proof.

1. $\underbrace{\frac{1}{2\pi} \int_T K_n(t) f_t dt}_{\text{B-valued}} = \underbrace{K_n * f}_{L^1-valued}$
2. $\lim_{n \to \infty} \frac{1}{2\pi} \int_T K_n(t)\varphi(t) dt = \varphi(0)$, for all continuous $\varphi : T \to B$
3. $\varphi : T \to B$, $\varphi(t) = f_t$ is continuous $\forall f \in B$
4. $\|K_n * f - f\|_B \to 0$

$\square$

Remark.

1. B norm Banach Space. Taking $K_n = F_n$ we have
$$\|\sigma_n(f) - f\|_B \to 0$$
for all $f \in B$.
2. Taking $B = L^p(T)$
   (a) $\|\sigma_n(f) - f\|_p \to 0$
   (b) $\overline{Trig(T)} = L^p(T)$

Remark. In $L^2(T)$

1. $\overline{Trig(T)} = L^2(T)$
2. $\overline{\mathrm{Span}\{\mathrm{e^{inx} : n \in \mathbb{Z}}\}} = L^2(T)$
3. $\{e^{inx} : n \in \mathbb{Z}\}$ ONB
4. Let the above ONB be written as $\{v_1, v_2, \ldots\}$, for all $f \in L^2(T)$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \langle f, v_i \rangle \, v_i = f$$
5. If $v = e^{ikx}$,
$$\langle f, v \rangle \, v = \left(\frac{1}{2\pi} \int_T f(x) e^{-ikx} dx \right) e^{ikx} = \left\langle f, e^{ikx} \right\rangle e^{ikx}$$
6. $\forall f \in L^2(T)$,
$$\|S_n(f) - f\|_2 \to 0$$