A Value Function Algorithm for the Stable Set Problem

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Contributions

- Rounding scheme for Lovász theta-function.
- Works on triangle-free perfect graphs and chordal graphs.
- No need to solve Lovász theta-function sequentially.

Maximum Stable Set Problem and Value Function Approximation.

- Maximum Stable Set Problem:

$$\max \sum_{i \in S} w_i \qquad \text{s.t.} \quad S \text{ is a stable set of } G,$$

where $G = (N, E)$ is a simple graph.
- $\mathcal{V} : 2^N \to \mathbb{R}_+$ is a *value function approximation* (VFA) if
  - it is monotone,
  - $\mathcal{V}(\emptyset) = 0$,
  - $\mathcal{V}(I) - \mathcal{V}(I \setminus (i \cup \delta(i))) \geq w_i, \quad \forall I \subseteq N, i \in I.$

Algorithm

Algorithm 1: Retrieving a stable set from a VFA

1. **Input:** $G = (N, E)$, a weight function w and a VFA $\mathcal{V}$.
2. $S := \emptyset$ ▷ Start with the empty stable set
3. $I := \{i \in N : \mathcal{V}(i) = w_i\}$ ▷ Discard all nodes believed to be suboptimal
4. **while** $I \neq \emptyset$ **do**
5. $\quad S := S \cup \{i\}$ for an arbitrary $i \in I$ ▷ Select i to join the stable set
6. $\quad I := I \setminus (\{i\} \cup \delta_i)$ ▷ Discard i and δ_i from remaining nodes
7. $\quad$ **for** $i \in I$ **do**
8. $\quad\quad$ **if** $\mathcal{V}(I) - \mathcal{V}(I \setminus (\{i\} \cup \delta_i) > w_i$ **then**
9. $\quad\quad\quad I := I \setminus \{i\}$. ▷ Discard i from remaining nodes
10. **Output:** Return S, a stable set of G.

LP Value Function

- Clique LP:

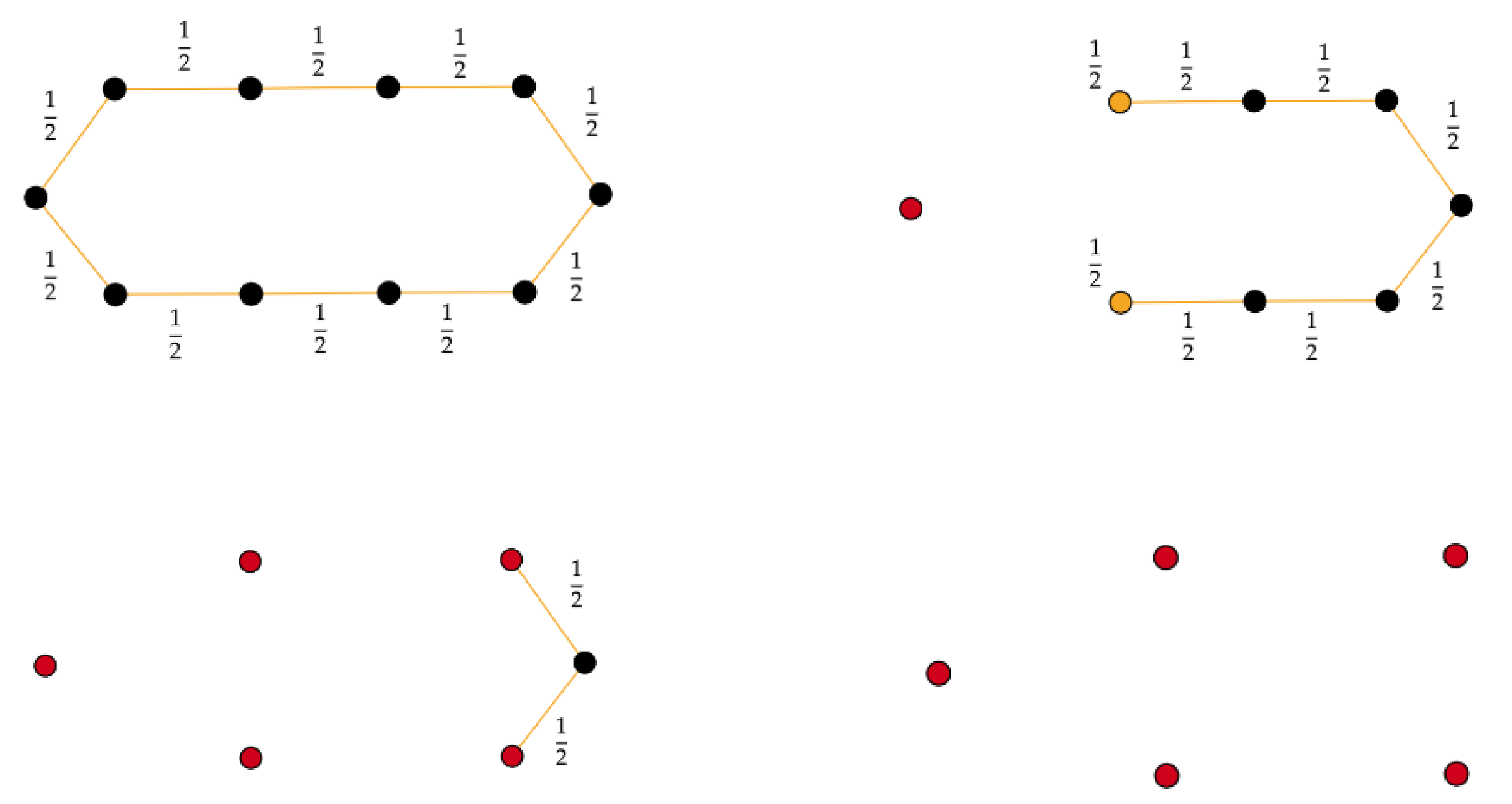
$$\max\ w^\top x \quad \text{s.t.} \sum_{i \in c} x_i \leq 1, \forall c \in C(G), \quad x \geq 0,$$

$$\min \sum_{c \in C} \mu_c \quad \text{s.t.} \sum_{c \ni i} \mu_c \geq w_i, \forall i \in N, \quad \mu_c \geq 0.$$

where $C(G)$ is the set of cliques contained in G.
- VFA based on LP: Given a pair of strictly complementary solutions (x^*, μ^*), for $I \subseteq N$,

$$V_{LP}(I) := \sum_{c \in C(G), c \cap I \neq \emptyset} \mu^*_c$$

Illustration of Algorithm 1 with V_{LP}



SDP Value Function

- Lovász-Schrijver SDP [1]:

$$\max_{x \in \mathbb{R}^n, X \in \mathbb{S}^n} \langle \mathrm{Diag}(w), X \rangle \quad \text{s.t.} \quad X_{ii} = x_i, \forall i \in N, \quad X_{ij} = 0, \forall ij \in E, \quad \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0, \qquad \text{(SDP-P)}$$

$$\min_{t,q,Q} t \quad \text{s.t.} \quad \begin{bmatrix} t & q^\top \\ q & Q \end{bmatrix} \succeq 0, \quad Q_{ij} := -2q_i - w_i, \forall i = j, \quad Q_{ij} = 0, \forall ij \notin E. \qquad \text{(SDP-D)}$$

- Given a relative-interior optimal solution (t, q, Q) for the dual problem (SDP-D),

$$V_{SDP}(I) \quad := \quad q_I^T \, Q_I^\dagger \, q_I,$$

where $Q^\dagger$ denotes the pseudo-inverse of Q.
- Algorithm 1 evaluates the function V_{SDP} at most $O(n)$ times, and produces a stable set $S \subseteq N$ for an arbitrary graph G. Notice that q, Q are fixed; only one SDP is solved, unlike many existing algorithms [2]–[4].

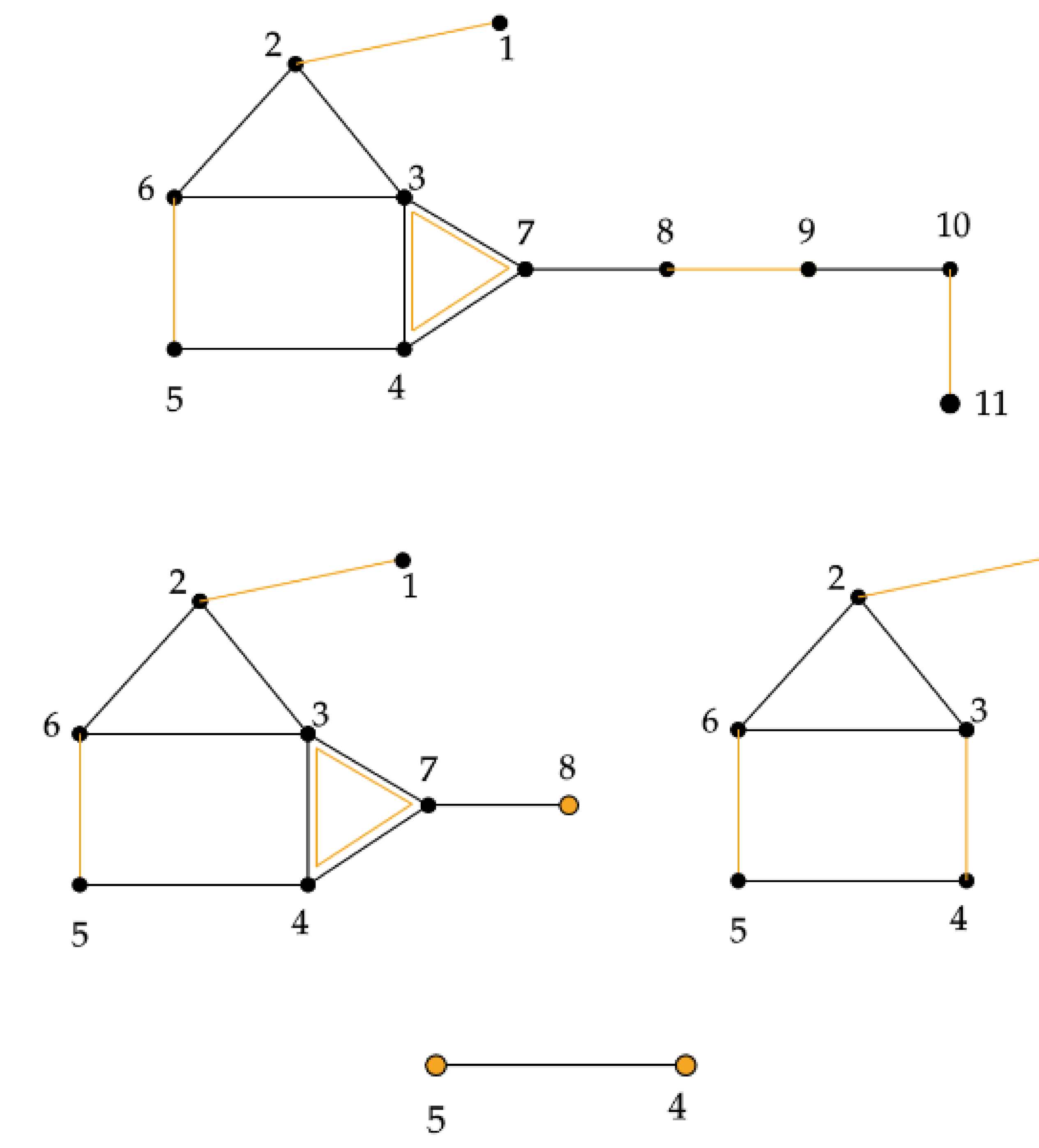
Theorem 1

Algorithm 1 with V_{LP} outputs a maximum stable set for all triangle-free perfect graphs and chordal graphs.

Theorem 2

Algorithm 1 with V_{SDP} is at least as good as the one with V_{LP}. In particular, it outputs a maximum stable set for triangle-free perfect graphs and chordal graphs.

Failing Example



References

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[2] F. Alizadeh, "A sublinear-time randomized parallel algorithm for the maximum clique problem in perfect graphs," in *Proceedings of the Second Annual ACM-SIAM Symposium on Discrete Algorithms*, ser. SODA '91, San Francisco, California, USA: Society for Industrial and Applied Mathematics, 1991, pp. 188–194.
[3] E. A. Yildirim and X. Fan-Orzechowski, "On Extracting Maximum Stable Sets in Perfect Graphs Using Lovász's Theta Function," *Computational Optimization and Applications*, vol. 33, no. 2, pp. 229–247, 2006.
[4] M. Grötschel, L. Lovász, and A. Schrijver, "Polynomial algorithms for perfect graphs," in *Topics on Perfect Graphs*, ser. North-Holland Mathematics Studies, vol. 88, North-Holland, 1984, pp. 325–356.